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Kneser's theorem and generalizations of Mann's theorem

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1 Introduction

One of the central questions in additive combinatorics concerns the growth of sumsets. In particular, one might wonder, for $A, B \subseteq \mathbb{N}$, how *large* the sumset $A + B = \{a + b : a \in A, b \in B\}$ is compared to the *size* of A and B.

Intuitively, if A and B are "random" or unstructured sets, we expect the sumset to be much larger than the individual sets. In particular, we will convince ourselves that the *density* of the sum of *random* sets tends, very quickly, to one.

On the contrary, structured sets may grow much more slowly, or even not at all. For instance, if we take the even numbers, $A = B = 2\mathbb{N}$, the sumset $A + B$ is still the original set. To work with these notions, we will need to formalize the meaning of *largeness.*

In what follows, we will look at two different notions of density, the first one is the *asymptotic lower density*, denoted $\underline{\mathrm{d}}(A)$, and the second one is the *Schnirelmann Density*, denoted $\sigma(A)$, both times for a set $A \subseteq \mathbb{N}$. Regarding the asymptotic lower density, Martin Kneser showed in [Kne53] that for $A, B \subseteq \mathbb{N}$, either

$$\underline{\mathrm{d}}(A + B) \geq \min(1, \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)),$$

or $A + B$ is essentially periodic. We will present a proof of Kneser's theorem. As the proof has many moving parts, in Section 1.2 we provide a roadmap of the different steps and theorems involved in proving Kneser's theorem.

With regard to the Schnirelmann density, Henry B. Mann showed in [Man42] that for $A, B \subseteq \mathbb{N}$, with $0 \in A \cap B$, we have

$$\sigma(A + B) \geq \min(1, \sigma(A) + \sigma(B)).$$

We will explore how this result behaves in two-dimensions through the explorations of different ways to generalize this notion of density, different constraints on A and B, and through variation of the result. We will conclude by looking at what research has been done regarding generalizations of Mann's Theorem and what is still left to be proved.

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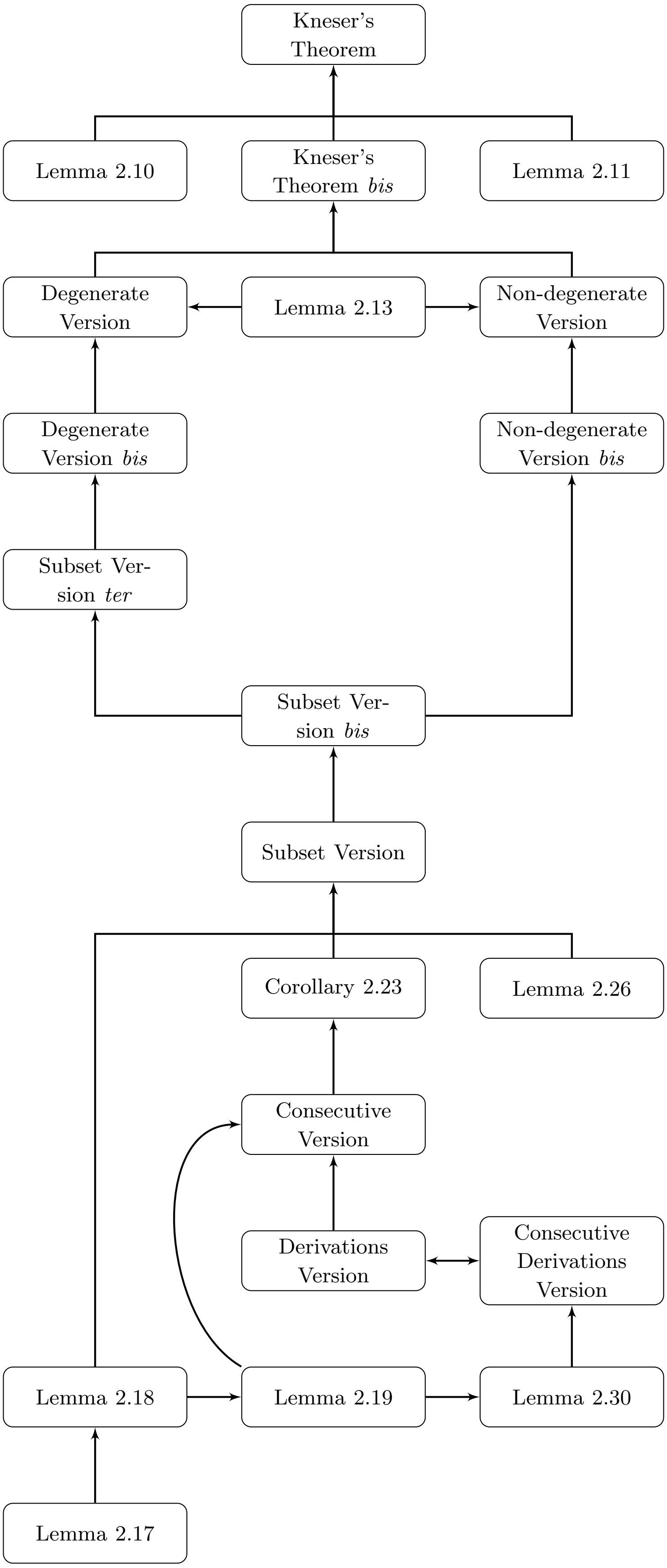
1.2 Roadmap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2 Kneser's theorem

2.1 Motivation and notation

In what follows, we denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$. For two sets $A, B \subseteq \mathbb{N}$, we define $A + B = \{a + b : a \in A, b \in B\}$. Similarly, for $t \in \mathbb{N}$, we write the set $\{t + a : a \in A\}$ as $A + t$ or $t + A$. We also write, for $k \in \mathbb{N}^*$, $kA = \{a_1 + \ldots + a_k : a_1, \ldots, a_k \in A\}$.

Additionally, we write $C(m, n) = |C \cap [m, n]|$ and $C(n) = C(1, n)$. Finally, we define the equivalence relation $A \sim B$ when two sets $A, B \subseteq \mathbb{N}$ are identical from some point onwards, by calling them *equivalent*.

Definition 2.1. Let $A \subseteq \mathbb{N}$. We define the *asymptotic lower density* of A as

$$\underline{\mathrm{d}}(A) := \liminf_{n\to\infty} \frac{A(n)}{n}.$$

Definition 2.1 encapsulates the notion of largeness for subsets of $\mathbb{N}$. It is not obvious how the lower-density of subsets grows under addition. For instance, we observe that, if $A = 2\mathbb{N}$, we have $2A = A$ and thus $\underline{\mathrm{d}}(A) = \underline{\mathrm{d}}(2A)$.

At the other extreme, if $S = \{n^2 : n \in \mathbb{N}\}$, we have $\underline{\mathrm{d}}(S) = 0$ and we know by Lagrange's theorem, [Lag72], that $4S = \mathbb{N}$ and so $\underline{\mathrm{d}}(4S) = 1$. To optimize this result, we use the Landau-Ramanujan theorem, [Lan08], which states that $2S(n) \asymp n/\sqrt{\log(n)}$. This result shows that $\underline{\mathrm{d}}(2S) = 0$, and thus setting $X = 2S$, we have found a subset of $\mathbb{N}$ such that $\underline{\mathrm{d}}(X) = 0$ and $\underline{\mathrm{d}}(2X) = 1$.

This difference naturally gives us the desire to understand how the lower-density of $A + B$ grows for $A, B \subseteq \mathbb{N}$. If A and B are "random" subsets, we expect the density of $A + B$ to be much bigger than the sum of the densities of A and B.

Indeed, suppose that we have two random subsets $A, B \subseteq \mathbb{N}$, with $\mathbb{P}(a \in A) = \alpha$ and $\mathbb{P}(b \in B) = \beta$, for $\alpha, \beta \in (0, 1)$ and every $a, b \in \mathbb{N}$. We thus have $\mathbb{E}[A(n)/n] = \alpha$, $\mathbb{E}[B(n)/n] = \beta$, and

$$\begin{aligned}\mathbb{E}[[A + B](n)/n] &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[k \in A + B] = \frac{1}{n}\sum_{k=1}^{n} 1 - \mathbb{E}[k \notin A + B] = 1 - \frac{1}{n}\sum_{k=1}^{n}(1 - \alpha\beta)^{k+1} \\ &= 1 - \left(\frac{1 - \alpha\beta}{\alpha\beta}\right)^2 \cdot \frac{1 - (1 - \alpha\beta)^n}{n} \xrightarrow{n\to\infty} 1.\end{aligned}$$

That said, we also saw the counterexample $A = B = 2\mathbb{N}$, and more generally, for any integer $k \geq 2$, taking $A = B = k\mathbb{N}$ is a counterexample to the incorrect guess that

$$\underline{\mathrm{d}}(A + B) \geq \min(1, \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)). \tag{1}$$

This family of counterexamples, is however very structured. And in this sense, Martin Kneser made a remarkable discovery in 1953, [Kne53].

Theorem 2.2 (Kneser's Theorem)**.** *Let $A, B \subseteq \mathbb{N}$. Either we have $\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B)$ or there must exist an integer $k \in \mathbb{N}$ such that for $H := k\mathbb{N}$ we have $(A + B + H)\backslash(A + B)$ is finite and*

$$\underline{d}(A + B) = \underline{d}(A + H) + \underline{d}(B + H) - \underline{d}(H).$$

The bound $\underline{\mathrm{d}}(A+B) \geq \underline{\mathrm{d}}(A)+\underline{\mathrm{d}}(B)$ is tight as we may find sets $A, B \subseteq \mathbb{N}$, with $\underline{\mathrm{d}}(A+B) = \underline{\mathrm{d}}(A)+\underline{\mathrm{d}}(B)$, such that there is no $k \in \mathbb{N}$ with $H = k\mathbb{N}$ and $(A + B + H)\backslash(A + B)$ finite.

Indeed, take an irrational $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We know by Weyl's Criterion, [Wey16], that $(\{n\alpha\})_{n=1}^{\infty}$ is equidistributed in $(0, 1)$. For $\theta_1, \theta_2 \in (0, 1)$ such that $\theta_1 + \theta_2 < 1$, we define $A = \{n : n \in \mathbb{N}, \{n\alpha\} \in [0, \theta_1]\}$ and $B = \{n : n \in \mathbb{N}, \{n\alpha\} \in [0, \theta_2]\}$. This gives $A + B \subseteq \{n : n \in \mathbb{N}, \{n\alpha\} \in [0, \theta_1 + \theta_2]\}$. Equidistribution implies that $\underline{\mathrm{d}}(A) = \theta_1$, $\underline{\mathrm{d}}(B) = \theta_2$, $\underline{\mathrm{d}}(A + B) \leq \theta_1 + \theta_2$, and that $A + B$ is not periodic from some point onward.

We will prove Theorem 2.2 (Kneser's Theorem) following Halberstam and Roth's sequences book from 1983, [HR83]. Instead of proving Kneser's original result, about the lower density of $\underline{\mathrm{d}}(A_0 + \ldots + A_k)$ for arbitrary $k \in \mathbb{N}^*$, we will study the case $k = 1$, which is cleaner than the original result.

2.2 A different wording

Let us rewrite Theorem 2.2 (Kneser's Theorem) with a wording more suitable to our work below.

Definition 2.3. Let $A, B \subseteq \mathbb{N}$. We define a *worse* pair $A', B' \subseteq \mathbb{N}$ if it satisfies both of the following conditions:

(i) $A \subseteq A'$ and $B \subseteq B'$,

(ii) $A + B \sim A' + B'$.

We observe that $\underline{d}(A) + \underline{d}(B) \leq \underline{d}(A') + \underline{d}(B')$, using (i), and that $\underline{d}(A + B) = \underline{d}(A' + B')$, using (ii). Thus, Theorem 2.2 (Kneser's Theorem) still holds if we replace A and B by A' and B' respectively.

Lemma 2.4. *Let $A, B \subseteq \mathbb{N}$ and $a, b \in \mathbb{N}$. We define $A' = A - a$ and $B' = B - b$. Then, we have $\underline{d}(A + B) = \underline{d}(A' + B')$ and $\underline{d}(A) + \underline{d}(B) = \underline{d}(A') + \underline{d}(B')$.*

Proof. We see that $\underline{d}(A) = \underline{d}(A')$, as the shift by a makes zero difference when the liminf approaches infinity (the same applies to $\underline{d}(B')$). Similarly, $A' + B' = A + B - a - b$ and thus $\underline{d}(A' + B') = \underline{d}(A + B)$. □

Definition 2.5. Let $g \in \mathbb{N}^*$ and $A \subseteq \mathbb{N}$. If there exists $C_A \subseteq [0, g-1]$ such that $A = C_A + g\mathbb{N}$, we say that A is *degenerate modulo g*. When we say that a set is *degenerate*, we simply mean that there exists a g such that the set is degenerate modulo g.

We notice that if A and B are equivalent to sets degenerate modulo g, for some g, then $A + B$ is also equivalent to a set degenerate to this same g. Additionally, when dealing with degenerate subsets (for a fixed g), we can just consider addition modulo g.

Theorem 2.6 (Cauchy-Davenport)**.** *Let $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$. We have $|A + B| \geq \min(p, |A| + |B| - 1)$.*

The proof is quite elementary and can be found here [Dav35]. What interests us about Theorem 2.6 (Cauchy-Davenport) is that if we have A and B degenerate modulo p, we cannot hope to prove something better than $\underline{d}(A+B) \geq \underline{d}(A) + \underline{d}(B) - 1/p$. From this we might be tempted to deduce the following bound. Let $A, B \subseteq \mathbb{N}$ degenerate modulo g, then we have

$$\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{g}. \tag{2}$$

Unfortunately, this cannot hold, as if A and B are degenerate modulo g, then they are also degenerate modulo kg, for any $k \in \mathbb{N}^*$. This implies that g needs to be *minimal.*

To define what minimal means here, we remind ourselves that any pair (A, B) might be replaced by a *worse* pair, as defined in Definition 2.3. This combined with the idea that g should be as small as possible gives us the following theorem.

Theorem 2.7 (Degenerate Version)**.** *Let $g \in \mathbb{N}^*$ and $A, B \subseteq \mathbb{N}$ be degenerate modulo g. Then, there exists a minimal divisor g' of g and a pair $A', B' \subseteq \mathbb{N}$, degenerate modulo g' such that (A', B') is worse than (A, B) and*

$$\underline{d}(A' + B') \geq \underline{d}(A') + \underline{d}(B') - \frac{1}{g'}.$$

This deals with the degenerate case. As discussed, if the pair $A, B \subseteq \mathbb{N}$ has a worse pair which itself is degenerate, then we might apply Theorem 2.7 (Degenerate Version) to the worse pair $A', B' \subseteq \mathbb{N}$, degenerate modulo g'. As seen, we have $\underline{d}(A + B) = \underline{d}(A' + B')$ and $\underline{d}(A) + \underline{d}(B) \leq \underline{d}(A') + \underline{d}(B')$, which gives us

$$\begin{aligned}
\underline{d}(A' + B') &\geq \underline{d}(A') + \underline{d}(B') - \frac{1}{g'} \\
\underline{d}(A + B) &\geq \underline{d}(A') + \underline{d}(B') - \frac{1}{g'} + (\underline{d}(A) + \underline{d}(B)) - (\underline{d}(A) + \underline{d}(B)) \\
&\geq \underline{d}(A) + \underline{d}(B) - \frac{1}{g'} + (\underline{d}(A') + \underline{d}(B') - \underline{d}(A) + \underline{d}(B)).
\end{aligned}$$

This result is even better than Theorem 2.7 (Degenerate Version), so Theorem 2.7 (Degenerate Version) still holds for pairs $A, B \subseteq \mathbb{N}$ such that there exists a worse degenerate pair $A', B' \subseteq \mathbb{N}$.

We would like to combine this with the following theorem.

Theorem 2.8 (Non-degenerate Version)**.** *Let $A, B \subseteq \mathbb{N}$. If (A, B) has no worse degenerate pair, including the pair (A, B) itself, then we have*

$$\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B).$$

We now combine Theorem 2.7 (Degenerate Version) and Theorem 2.8 (Non-degenerate Version) to get:

Theorem 2.9 (Kneser's Theorem *bis*)**.** *Let $A, B \subseteq \mathbb{N}$. Either $\underline{d}(A+B) \geq \underline{d}(A)+\underline{d}(B)$ or there exists $g' \in \mathbb{N}^*$ and a pair $A', B' \subseteq \mathbb{N}$ degenerate modulo g' such that (A', B') is worse than (A, B) and*

$$\underline{d}(A' + B') \geq \underline{d}(A') + \underline{d}(B') - \frac{1}{g'}.$$

Let us now see how Theorem 2.9 (Kneser's Theorem *bis*) implies Theorem 2.2 (Kneser's Theorem). As we would like a degenerate pair modulo g', we take $H = g'\mathbb{N}$.

Lemma 2.10. *Let $g' \in \mathbb{N}$, $A, B \subseteq \mathbb{N}$, $A', B' \subseteq \mathbb{N}$ degenerate modulo g', such that (A', B') is worse than (A, B), and set $H = g'\mathbb{N}$. We have that $(A + B + H)\backslash(A + B)$ is finite.*

Proof. Let us note that as A' and B' are degenerate modulo g', we have that $A' + B' + H = A' + B'$. And as $A' + B' \sim A + B$ we have that $(A' + B' + H)\backslash(A + B)$ is finite. We conclude by using that $A \subseteq A'$ and $B \subseteq B'$ and so $(A + B + H) \subseteq (A' + B' + H)$ which shows that $(A + B + H)\backslash(A + B)$ is finite. □

Lemma 2.11. *Let $g' \in \mathbb{N}$, $A, B \subseteq \mathbb{N}$, $A', B' \subseteq \mathbb{N}$ degenerate modulo g', such that (A', B') is worse than (A, B), and set $H = g'\mathbb{N}$. We have that*

$$\underline{d}(A + B) = \underline{d}(A + H) + \underline{d}(B + H) - \underline{d}(H).$$

Proof. We suppose that Theorem 2.9 (Kneser's Theorem *bis*) is true. Using that $A + B \sim A' + B'$, and then that A' and B' are degenerate, we thus have

$$\begin{aligned} \underline{\mathrm{d}}(A' + B') &\geq \underline{\mathrm{d}}(A') + \underline{\mathrm{d}}(B') - \frac{1}{g'} \\ \underline{\mathrm{d}}(A + B) &\geq \underline{\mathrm{d}}(A') + \underline{\mathrm{d}}(B') - \underline{\mathrm{d}}(H) \\ \underline{\mathrm{d}}(A + B) &\geq \underline{\mathrm{d}}(A + H) + \underline{\mathrm{d}}(B + H) - \underline{\mathrm{d}}(H) \end{aligned}$$

On the other side, we know that $\underline{\mathrm{d}}(A + B) < \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$. So $\underline{\mathrm{d}}(A + B)$ is trapped inside an interval of size at most $1/g'$. However, using the fact that $A + B \sim A' + B'$, which is degenerate modulo g', we know any change in $C_{A'+B'}$ as defined in Definition 2.3, will change $\underline{\mathrm{d}}(A' + B')$ by at least $1/g'$. So we must have $\underline{\mathrm{d}}(A + B) = \underline{\mathrm{d}}(A + H) + \underline{\mathrm{d}}(B + H) - \underline{\mathrm{d}}(H)$. □

Lemma 2.10 and Lemma 2.11 use the appropriate language to show that Theorem 2.9 (Kneser's Theorem *bis*) implies Theorem 2.2 (Kneser's Theorem).

2.3 Some equivalent theorems

Proving Theorem 2.9 (Kneser's Theorem *bis*) is far from obvious. Before going any further, we need to introduce some more notation.

Definition 2.12. Given $g \in \mathbb{N}^*$ and $A \subseteq \mathbb{N}$ we define $A^{(g)} = \{a + gz : a \in A, z \in \mathbb{Z}\} \cap \mathbb{N}$.

We remark that $A^{(g)}$ is the minimal set containing A which is degenerate modulo g. We then deduce a few properties.

Lemma 2.13. *Let $A, B \subseteq \mathbb{N}$ and $g, h \in \mathbb{N}^*$. Write $d = \gcd(g, h)$. The following properties apply:*

(i) we have ${A^{(g)}}^{(h)} = A^{(d)}$,

(ii) we have that $(A + B)^{(g)} = A^{(g)} + B^{(g)}$,

(iii) if $A^{(g)} = A^{(h)}$, we have $A^{(g)} = A^{(h)} = A^{(d)}$.

Proof. (i) We write ${A^{(g)}}^{(h)} = \{a + gz + hz' : a \in A, z, z' \in \mathbb{Z}\} \cap \mathbb{N}$. However, numbers of the form $gz + hz'$ consist of multiples of d. (ii) We write $(A + B)^{(g)} = \{a + b + gz : a \in A, b \in B, z \in \mathbb{Z}\} = \{a + b + gz + gz' : a \in A, b \in B, z, z' \in \mathbb{Z}\} = A^{(g)} + B^{(g)}$. (iii) We use (i) to show that $A^{(d)} = {A^{(g)}}^{(h)} = {A^{(h)}}^{(h)} = A^{(h)} = A^{(g)}$.

□

We are now ready to state equivalent forms of Theorem 2.7 (Degenerate Version) and Theorem 2.8 (Non-degenerate Version) respectively.

Theorem 2.14 (Degenerate Version *bis*)**.** *Let $A, B \subseteq \mathbb{N}$ be degenerate modulo $g \in \mathbb{N}$. Then, there exists a minimal divisor g' of g such that $(A + B)^{(g')} = A + B$ and that*

$$\underline{d}(A^{(g')} + B^{(g')}) \geq \underline{d}(A^{(g')}) + \underline{d}(B^{(g')}) - \frac{1}{g'}.$$

Theorem 2.15 (Non-degenerate Version *bis*)**.** *Let $A, B \subseteq \mathbb{N}$ such that $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$. Then, there exists $g \in \mathbb{N}^*$ such that $A + B \sim (A + B)^{(g)}$.*

We use Lemma 2.13 to show that Theorem 2.14 (Degenerate Version *bis*) implies Theorem 2.7 (Degenerate Version). Indeed, the pair $(A^{(g')}, B^{(g')})$ is worse than (A, B). Similarly, Theorem 2.15 (Non-degenerate Version *bis*) implies Theorem 2.8 (Non-degenerate Version) as the pair $(A^{(g)}, B^{(g)})$ is worse than (A, B).

Conversely, let us see how Theorem 2.7 (Degenerate Version) implies Theorem 2.14 (Degenerate Version *bis*). By Theorem 2.7 (Degenerate Version), for $g \in \mathbb{N}^*$ and the pair $A, B \subseteq \mathbb{N}$ degenerate modulo g, there exists a divisor g' of g and a worse pair $A', B' \subseteq \mathbb{N}$ degenerate modulo g' such that $\underline{d}(A' + B') \geq \underline{d}(A') + \underline{d}(B') - 1/g'$. Here, the pair (A', B') is also worse than the pair $(A^{(g')}, B^{(g')})$, and in particular $A + B \sim (A + B)^{(g')}$.

We actually even have that $A + B = (A + B)^{(g')}$, as if there exists an $x \in (A + B)^{(g')} \backslash (A + B)$, it would contain the entire congruence class modulo g, which would not be in $A + B$, contradicting $A + B \sim (A + B)^{(g')}$. Additionally, we get $\underline{d}((A + B)^{(g')}) \geq \underline{d}(A^{(g')}) + \underline{d}(B^{(g')}) - 1/g'$, showing that Theorem 2.7 (Degenerate Version) implies Theorem 2.14 (Degenerate Version *bis*).

To see that Theorem 2.8 (Non-degenerate Version) implies Theorem 2.15 (Non-degenerate Version *bis*), we take $A, B \subseteq \mathbb{N}$ such that $\underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$. By Theorem 2.8 (Non-degenerate Version), there must exist a $g \in \mathbb{N}^*$ and a worse pair $A', B' \subseteq \mathbb{N}$ such that A' and B' are degenerate modulo g. This means that $A + B \subseteq (A + B)^{(g)} = A^{(g)} + B^{(g)} \subseteq A' + B'$. Using that $A + B \sim A' + B'$, we are able to conclude that $A + B \sim (A + B)^{(g)}$. We have now showed that Theorem 2.8 (Non-degenerate Version) implies Theorem 2.15 (Non-degenerate Version *bis*).

The proofs of these two theorems are still far from easy. To provide a proof, we still need a few tools and theorems. The following theorem has the essence of Kneser's theorem. We will prove it in the following two subsections.

Theorem 2.16 (Subset Version)**.** *Let $A, B \subseteq \mathbb{N}$ such that $0 \in A \cap B$ and $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$. Then, there must exist an integer $g \in \mathbb{N}^*$ and a subset $C \subseteq A + B$, such that $0 \in C$, $C \sim C^{(g)}$ and*

$$\underline{d}(C) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{g}.$$

2.4 Working with τ-transformations

To prove Theorem 2.16 (Subset Version), we will repeatedly transform A and B to make them more structured. We denote for an element $a_* \in A$ the τ-transformation $(A,B)^\tau := (A^\tau, B^\tau)$, where $\tau = \tau(a_*)$, by

$$A^\tau = A \cup (B + a_*) \qquad \text{and} \qquad B^\tau = B \cap (A - a_*). \tag{3}$$

This transformation has some very useful properties that we will use throughout the rest of the proof of Theorem 2.16 (Subset Version).

Lemma 2.17. *Let $A, B \subseteq \mathbb{N}$ and $a_* \in A$. The τ-transform $\tau = \tau(a_*)$ has the following properties.*

(i) $A \subseteq A^\tau$ *and* $B^\tau \subseteq B$

(ii) $A^\tau + B^\tau \subseteq A + B$

(iii) $B^\tau + a_* \subseteq A \subseteq A^\tau$

(iv) if $0 \in B$, *then we also have* $0 \in B^\tau$

(v) $\underline{d}(A^\tau) + \underline{d}(B^\tau) = \underline{d}(A) + \underline{d}(B)$

Proof. The properties (i), (iii), and (iv) are direct from the definition. To show (ii), we observe that for $b \in B$ and $b^\tau \in B^\tau$, we can rewrite $(a_* + b) + b^\tau$ as $(a_* + b^\tau) + b \in A + B$ using (iii). To show (v), we observe that we removed a certain subset B' from B. This subset is exactly $B \backslash (A - a_*)$. Thus the subset $B' + a_*$ corresponds to $(B + a_*) \backslash A$, which is exactly what we add to A. In other words, $B^\tau = B \backslash B''$ and $A^\tau = A \oplus (B' + a_*)$, where $\oplus$ is a disjoint union.

From this, we gather that $A^\tau(n) + B^\tau(n)$ differ by at most a_* from $A(n) + B(n)$, so we indeed have $\underline{d}(A^\tau) + \underline{d}(B^\tau) = \underline{d}(A) + \underline{d}(B)$. ☐

We will sometimes need to apply multiple τ-transformations in a row. So, for $a_0 \in A$ and $a_1 \in A^{\tau_0}$, we write $\tau(a_0) = \tau_0$ and $\tau(a_1) = \tau_1$ as well as $(A,B)^{\tau_0\tau_1} = ((A,B)^{\tau_0})^{\tau_1}$ meaning that we transform $(A,B)^{\tau_0}$ with τ_1. When we write $(A,B)^T$, we see a *derivation* T as a finite succession of τ-transformation. That is $(A,B)^T = (A,B)^{\tau_1\tau_2\cdots\tau_k}$ for some arbitrary $\tau_1 \ldots \tau_k$.

Lemma 2.18. *Let $A, B \subseteq \mathbb{N}$ and some $T = \tau_1 \ldots \tau_k$. We have the following properties:*

(i) $A \subseteq A^T$ *and* $B^T \subseteq B$,

(ii) $A^\tau + B^\tau \subseteq A + B$,

(iii) if $0 \in B$, *then* $0 \in B^T$,

(iv) $\underline{d}(A^T) + \underline{d}(B^T) = \underline{d}(A) + \underline{d}(B)$.

Proof. All these properties follow directly from Lemma 2.17. ☐

From these we obtain the following result.

Lemma 2.19. *Let $A, B \subseteq \mathbb{N}$, $S \subseteq A$ a finite subset and $0 \in A \cap B$. Then, there exists a derivation T of (A,B) such that $S + B^T \subseteq A^T$.*

Proof. Suppose that $S = \{a_1, \ldots, a_n\} \subseteq A$. If $\tau_1 = \tau(a_1)$ is applied to (A,B), we have by (iii) of Lemma 2.17 that $B^{\tau_1} + a_1 \subseteq A^{\tau_1}$. Similarly, if $\tau_2 = \tau(a_2)$ applied to $(A,B)^{\tau_1}$, we have that. $B^{\tau_1\tau_2} + a_2 \subseteq A^{\tau_1\tau_2}$. Now, using (i) of Lemma 2.17 we have $a_1 + B^{\tau_1\tau_2} \subseteq a_1 + B^{\tau_1} \subseteq A^{\tau_1} \subseteq A^{\tau_1\tau_2}$, implying that $\{a_1, a_2\} + B^{\tau_1\tau_2} \subseteq A^{\tau_1\tau_2}$.

We proceed like this by induction. Suppose that $\{a_1, \ldots, a_{k-1}\} + B^{\tau_1\cdots\tau_{k-1}} \subseteq A^{\tau_1\cdots\tau_{k-1}}$, where $\tau_i = \tau(a_i)$ for all $i \in [n]$. We apply τ_k to $(A,B)^{\tau_1\cdots\tau_{k-1}}$, which gives us $a_k + B^{\tau_1\cdots\tau_k} \subseteq A^{\tau_1\cdots\tau_k}$ and for $l < k$ we have

$$a_l + B^{\tau_1\cdots\tau_k} \subseteq a_l + B^{\tau_1\cdots\tau_{k-1}} \subseteq \ldots \subseteq a_l + B^{\tau_1\cdots\tau_l} \subseteq A^{\tau_1\cdots\tau_l} \subseteq \ldots \subseteq A^{\tau_1\cdots\tau_k},$$

which gives us, when $k = n$ and $T = \tau_1 \ldots \tau_n$, that $S + B^T \subseteq A^T$. ☐

We note, using (ii), (iii), and (iv) of Lemma 2.18, that every derivation T of (A, B) satisfies the hypothesis of Theorem 2.16 (Subset Version) and that if the conclusion of Theorem 2.16 (Subset Version) holds for a particular derivation T of (A, B), then the conclusions also hold for (A, B). We thus get that the conclusions of Theorem 2.16 (Subset Version) hold if the corresponding statement, with A and B replaced by A^T and B^T, holds for some T.

The goal is now to find a suitable derivation T for which Theorem 2.16 (Subset Version) is easier to prove. Indeed, we transform Theorem 2.16 (Subset Version) into Theorem 2.22 (Consecutive Version) with a suitable derivation. Before going there, we will need a technical lemma.

For two sets $A, B \subseteq \mathbb{N}$, we write the multiset $A \vee B$ as the union of A and B counted with multiplicity. For instance, $\{1, 2, 3\} \vee \{2, 3, 4\} = \langle 1, 2, 2, 3, 3, 4\rangle$.

Definition 2.20. Let $C \subseteq \mathbb{N}$. We say that an interval $[m, n]$ is C-good with respect to η if

$$C(m, n) \geq \eta \cdot (n - m + 1).$$

Lemma 2.21. *Let $\theta \in \{0, 1\}$, $\eta \in (0, 1]$, $n \in \mathbb{N}^*$, and $A, B \subseteq \mathbb{N} \cap [0, n]$ with $0 \in A \cap B$. Then, we have that*

$$[\theta, m] \text{ is } (A \vee B)\text{-good with respect to } \eta \text{ for } m = \theta, \theta + 1, \ldots, n, \tag{4}$$

implies

$$[\theta, m] \text{ is } (A + B)\text{-good with respect to } \eta \text{ for } m = \theta, \theta + 1, \ldots, n, \tag{5}$$

Proof. The proof is lengthy and does not directly relate to what we are proving, so it can be found in the Appendix, Lemma A.1. □

Theorem 2.22 (Consecutive Version)**.** *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$. If A contains $m \in \mathbb{N}^*$ consecutive integers and if $\underline{d}(A + B) < \frac{m}{m+1}(\underline{d}(A) + \underline{d}(B))$, then we have $A + B \sim \mathbb{N}$.*

Proof. Let $a, a + 1, \ldots, a + m - 1$ be the m consecutive elements in A. We start by shifting A by a, so $A' = A - a$, we still have $0 \in A \cap B$ and by Lemma 2.4 we have $\underline{d}(A' + B) = \underline{d}(A + B)$ and $\underline{d}(A') + \underline{d}(B) = \underline{d}(A) + \underline{d}(B)$. Meaning the result is invariant under the translation of A by a. This allows us to suppose, without loss of generality, that $\{0, 1, \ldots, m - 1\} \subseteq A$. Additionally, using Lemma 2.19, we can even find a derivation T such that

$$\{0, 1, \ldots, m - 1\} + B^T \subseteq A^T.$$

Indeed, if we have a derivation T such that $A^T + B^T \sim \mathbb{N}$, then using (ii) of Lemma 2.18, we have $A + B \sim \mathbb{N}$ as well.

Now, we fix γ a positive number satisfying $\gamma < \underline{d}(A) + \underline{d}(B)$, and in particular we set

$$\gamma = \frac{m+1}{m} \quad \text{if} \quad \frac{m}{m+1}(\underline{d}(A) + \underline{d}(B)) > 1.$$

This implies that we always have $\frac{m}{m+1}\gamma \leq 1$. Having $\gamma < \underline{d}(A) + \underline{d}(B)$ also implies that there is a $x_0 \in \mathbb{R}$, such that $A(x) + B(x) \geq \gamma x$ for all $x \geq x_0$. We set x_0 to be the least positive integer satisfying it. This choice of x_0 implies that $A(x_0 - 1) + B(x_0 - 1) \leq \gamma(x_0 - 1)$. Additionally, we can subtract $A(x) + B(x) \geq \gamma x$ from it and replace x by $x + x_0$ to get

$$A(x_0, x_0 + x) + B(x_0, x_0 + x) \geq \gamma(x + 1), \tag{6}$$

for all $x \in \mathbb{N}$. In other words, we showed that $[x_0, x_0 + x]$ is $(A \vee B)$-good with respect to γ for $x \in \mathbb{N}$.

If we set $x = 0$ in (6), we get that $x_0 \in A$ or $x_0 \in B$. However, we assumed that $B \subseteq A$ so we have $x_0 \in A$. Now, we set x_1 to be the smallest element of B no smaller than x_0. Using this, we define $A' = (A - x_0) \cap \mathbb{N}$ and $B' = (B - x_1) \cap \mathbb{N}$. The goal will be to show that

$$[0, x] \text{ is } (A' \vee B')\text{-good with respect to } \frac{m}{m+1}\gamma \text{ for } x = 0, 1, 2, \ldots$$

Using Lemma 2.21, it will follow that

$$[0, x] \text{ is } (A' + B')\text{-good with respect to } \frac{m}{m+1}\gamma \text{ for } x = 0, 1, 2, \ldots,$$

which will allow us to conclude. We know that $[x_0, x_1]$ contains exactly one element of B. Thus we have

$$B'(x) = B(x_1 + 1, x_1 + x) = B(x_0, x + 1 + x) - 1 \geq B(x_0, x_0 + x) - 1,$$

and similarly for A, we have

$$A'(x) = A(x_0 + 1, x_0 + x) = A(x_0, x_0 + x) - 1 \geq A(x_0, x_0 + x) - 1.$$

It then follows from (6) that

$$1 + A'(x) + B'(x) \geq \gamma(x + 1) - 1 \geq \frac{m}{m+1}\gamma(x + 1),$$

for all $x \geq \frac{(m+1)}{\gamma} - 1$. We have thus proved that $[0, x]$ is $(A' \vee B')$-good with respect to $\frac{m}{m+1}\gamma$ for $x \geq \frac{(m+1)}{\gamma} - 1$. To finish, we need to show that it also holds for $x < \frac{(m+1)}{\gamma} - 1$.

If $0 \leq x < x_1 - x_0$, we have $[x_0, x_0 + x] \cap B = \emptyset$. So using (6), we see that $[x_0, x_0 + x]$ is A-good with respect to γ and therefore $[0, x]$ is A'-good with respect to γ. We are left with the case $x_1 - x_0 \leq x < \frac{m+1}{\gamma} - 1$. Let us use that $x_1 + \{0, 1, \ldots, m - 1\} \subseteq A$. If $x < x_1 - x_0 + m$, we write $[0, x] = [0, x_1 - x_0 - 1] \cup [x_1 - x_0, x]$. We already showed that $[0, x_1 - x_0 - 1]$ is A'-good with respect to γ, and $[x_1 - x_0, x]$ is contained in A' so is A'-good with respect to $\frac{m}{m+1}\gamma$. Thus, when $x < x_1 - x_0 + m$ we have that $[0, x]$ is A'-good with respect to $\frac{m}{m+1}\gamma$.

We are now left with the case $x_1 - x_0 + m \leq x < \frac{m+1}{\gamma} - 1$. We know that $[0, x]$ contains at least m elements of A' and, by rearranging $x < \frac{m+1}{\gamma} - 1$, we have $m > \frac{m}{m+1}\gamma(x + 1)$. So we have showed that $[0, x]$ is $(A' + B')$-good with respect to $\frac{m}{m+1}\gamma$ for $x = 0, 1, 2, \ldots$.

We now conclude. Note that $\frac{m}{m+1}\left(\underline{d}(A) + \underline{d}(B)\right) > 1$. Indeed, if $\frac{m}{m+1}\left(\underline{d}(A) + \underline{d}(B)\right) \leq 1$, we have $\underline{d}(A + B) = \underline{d}(A' + B') \geq \frac{m}{m+1}\gamma$ for any $\gamma \leq \underline{d}(A) + \underline{d}(B)$, and thus $\underline{d}(A + B) \geq \frac{m}{m+1}(\underline{d}(A) + \underline{d}(B))$ contradicting the hypothesis.

As we wrote $\gamma = \frac{m+1}{m}$, if $\frac{m}{m+1}(\underline{d}(A) + \underline{d}(B))$, we have $\frac{m}{m+1}\gamma = 1$ and so we have, for $x \in \mathbb{N}$, $[0, x]$ is $(A' + B')$-good with respect to 1, and hence $A' + B' \sim \mathbb{N}$. Additionally, $A' + B'(x_0 + x_1) = (A' + x_0) + (B' + x_1) \subseteq A + B$ and thus $A + B \sim \mathbb{N}$. □

Corollary 2.23. *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$ and $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$. Then, if there exists a sequence of natural numbers $(m_i)_{i=1}^{\infty}$ such that $m_i \to \infty$ as $i \to \infty$ and corresponding derivations $\left((A, B)^{T_i}\right)_{i=0}^{\infty}$ such that A^{T_i} contains m_i consecutive integers, it follows that $A + B \sim \mathbb{N}$.*

Proof. Since $\frac{m_i}{m_i+1} \to 1$ as $i \to \infty$ and $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$, there exists $r \in \mathbb{N}$ such that

$$\underline{d}(A + B) < \frac{m_r}{m_r + 1}(\underline{d}(A) + \underline{d}(B)),$$

and this result follows from (ii) and (iv) from Lemma 2.18 combined with Theorem 2.22 (Consecutive Version). □

2.5 Proof of Subset Version

We mentioned that the goal is to find a suitable derivation of (A, B) for which Theorem 2.16 (Subset Version) would reduce to Theorem 2.22 (Consecutive Version). To do so, we introduce two particular functions.

Definition 2.24. Let $A \subseteq \mathbb{N}$. We define $f(A)$ to be the smallest positive difference of a pair of elements of A. That is,

$$f(A) = \min_{\substack{s, s' \in A \\ s \neq s'}} |s - s'|.$$

Definition 2.25. Let $A \subseteq \mathbb{N}$. We define $g(A)$ to be the greatest common divisor of the elements of A.

For our purposes, we always assume that $0 \in A$. In particular, if $A = \{0\}$, we set $f(A) = g(A) = \infty$. It is clear that $g(A)$ divides $f(A)$ and that $g(A) \leq f(A)$ for all $A \subseteq \mathbb{N}$.

Lemma 2.26. *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$. For every derivation T of (A, B), we have*

$$\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{f(B^T)}, \tag{7}$$

and

$$\underline{d}(A^T + B^T) \geq \underline{d}(A^T) + \underline{d}(B^T) - \frac{1}{g(B^T)}. \tag{8}$$

Proof. First, for any derivation T, we have $0 \in B^T$. We now set $b_0 = 0$ and b_n to be the n-th element of B^T ordered from smallest to largest. We thus have for $r \in \mathbb{N}^*$ that $b_r = \sum_{n=1}^{r}(b_n - b_{n-1}) \geq r \cdot f(B^T)$, and if we take b_r to be the largest element of $B^T \cap [0, x]$, we get that $B^T(x) = B^T(u_r) = r \leq u_r / f(B^T) \leq x / f(B^T)$. Hence we have

$$\underline{d}(A^T + B^T) \geq \underline{d}(A^T) \geq \liminf_{x \to \infty} \frac{1}{x}\left(A^T(x) + B^T(x) - \frac{x}{f(B^T)}\right) \geq \underline{d}(A^T) + \underline{d}(B^T) - \frac{1}{f(B^T)}.$$

Thus, (7) follows from Lemma 2.18, and (8) follows from (7) and $g(A) \leq f(A)$. ☐

If the set $\{f(B^T)\}$, of all possible derivations T of (A, B), is unbounded, then Lemma 2.26 tells us $\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B)$, which contradicts the hypothesis of Theorem 2.16 (Subset Version). So in what follows, we will assume that

$$\text{the set } \{f(B^T)\} \text{ is bounded,} \tag{9}$$

and thus that

$$\text{the set } \{g(B^T)\} \text{ is bounded.} \tag{10}$$

Let $g = \max_T g(B^T)$. We will prove Theorem 2.16 (Subset Version) for this g and for $C = A^T + B^T$, where T is a derivation such that $g(B^T) = g$. Combining Lemma 2.18 and Lemma 2.26 we get that the statement

"*Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$. If $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$ and if the derivations of A and B satisfy (10), we have a derivation T of A and B with $g(B^T) = g$ such that $A^T + B^T \sim (A^T + B^T)^{(g)}$*"

is equivalent to Theorem 2.16 (Subset Version). We will even express this statement in a more convenient form, to do so let us see the following.

Lemma 2.27. *Let $A, B \subseteq \mathbb{N}$, for any derivations T of A and B, we have $f(B) \leq f(B^T)$ and $g(B) \leq g(B^T)$.*

Proof. We use Lemma 2.18 to observe that $B^T \subseteq B$ and thus the lemma follows from definition of f and g. ☐

Now, let T be some derivation that maximizes $g(B^T)$, that is $g(B^T) = g$. From Lemma 2.27, we have that for any derivation T' of A^T and B^T, we have $g(B^{TT'}) = g$. We can thus reduce the proof of Theorem 2.16 (Subset Version) to the following theorem.

Theorem 2.28 (Derivations Version)**.** *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$. If $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$ and if for every derivation T of A and B we have $g(B^T) = g(B) = g$, then $A + B \sim (A + B)^{(g)}$.*

In order to prove Theorem 2.28 (Derivations Version), we will prove a special case of this theorem, which turns out to be equivalent to Theorem 2.28 (Derivations Version).

Theorem 2.29 (Consecutive Derivations Version)**.** *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$. If $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$, if for every derivation T of A and B we have $g(B^T) = g(B) = g$, and if A contains g consecutive integers, then $A + B \sim \mathbb{N}$.*

Let us show that Theorem 2.28 (Derivations Version) and Theorem 2.29 (Consecutive Derivations Version) are equivalent.

Proof. Let $n \in \mathbb{N}^*$ be a natural number and let $\{s_0, s_1, \ldots, s_j\} \subseteq \mathbb{N}$ a set of j integers, incongruent modulo n. By an incongruent set, we mean that any two elements of the set are not congruent. The set $\{s_0, s_1, \ldots, s_j\}^{(n)}$ consists of the integers of the form $s_i + tn$, where $i \in \{0, 1, \ldots, j\}$ and $t \in \mathbb{N}$.

Suppose now that we have $n, n^* \in \mathbb{N}$, $\{r_0, \ldots, r_{h-1}\} \subseteq \mathbb{N}$ and $\{r_0^*, \ldots, r_{h-1}^*\} \subseteq \mathbb{N}$, h numbers incongruent modulo n and n^* respectively, as well as having $r_0 = r_0^* = 0$.

We define the map $\Omega : \{r_0, \ldots, r_{h-1}\}^{(n)} \to \{r_0^*, \ldots, r_{h-1}^*\}^{(n^*)}$ by $\Omega(r_i + tn) = r_i^* + tn^*$, for $i \in \{0, 1, \ldots, h-1\}$ and $t \in \mathbb{N}$. In particular we have $\Omega(r_i) = r_i^*$ and $\Omega(n) = n^*$. We will thus extend the * notation. We define $\Omega(x) = x^*$ for any $x \in \{r_0, \ldots, r_{h-1}\}^{(n)}$ and similarly for any subset $X \subseteq \{r_0, \ldots, r_{h-1}\}^{(n)}$ we write $\Omega(X) = X^*$.

The function Ω has the following properties that follow from the definition:

(i) $\frac{x}{n} = \frac{x^*}{n^*} + O(1)$ where x is the only variable.

(ii) If $X \subseteq \{0\}^{(n)}$, then $n^{-1}g(X) = (n^*)^{-1}g(X^*)$.

(iii) If a^* is defined and $x \in \{0\}^{(n)}$, then $(a + x)^* = a^* + x^*$.

(iv) If a and x both lie in $\{r\}^{(n)}$, thus $(x - a)^* = x^* - a^*$.

For what follows, we will suppose that we have $A, B \subseteq \mathbb{N}$ with $A \subseteq \{r_0, \ldots, r_{h-1}\}^{(n)}$ and $B \subseteq \{0\}^{(n)}$.

First, we show that for a transformation $\tau = \tau(a_0)$ of A and B, and applying to it the map Ω is the same as applying the map to A and B and then transform it by $\tau^* = \tau(a_0^*)$. More formally, this corresponds to $((A, B)^\tau)^* = (A^*, B^*)^{\tau^*}$. Indeed, by (iii) and (iv) we have

$$(a_0 + B)^* = a_0^* + B^* \qquad \text{and} \qquad (B \cap (A - a_0))^* = B^* \cap (A^* - a_0^*).$$

From this we deduce that every derivation T of A and B satisfies $\big((A, B)^T\big)^* = (A^*, B^*)^{T^*}$. Additionally, since $B \subseteq \{0\}^{(n)}$, (iii) gives us $(A + B)^* = A^* + B^*$. Using (i) and (ii) we have that, for any derivation T of A and B,

$$n\underline{\mathrm{d}}(A + B), \qquad n(\underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)), \qquad n^{-1}g(B^T)$$

are invariant under Ω.

Now, suppose that A and B satisfy the hypothesis of Theorem 2.28 (Derivations Version). That is $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$, $\underline{\mathrm{d}}(A + B) < \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B)$, and for every derivation T of A and B we have $g(B^T) = g(B) = g$. Then, take h to be the maximal number of elements of A incongruent modulo g. Using that $0 \in A$ and without loss of generality, we can suppose that $A \subseteq \{r_0, \ldots, r_h\}^{(g)}$, where $r_0 = 0$. We can ensure that $B^T \subseteq \{0\}^{(g)}$ as $g(B^T) = g(B) = g$.

Take, now, Ω as the mapping from $\{r_0, r_1, \ldots, r_{h-1}\}^{(g)} \to \{0, 1, \ldots, h-1\}^{(h)}$. Thus $\Omega(A)$ contains h consecutive integers and so $\Omega(A)$ and $\Omega(B)$ satisfy the hypothesis of Theorem 2.29 (Consecutive Derivations Version) replacing g with h. Suppose the Theorem 2.29 (Consecutive Derivations Version) holds for $\Omega(A)$ and $\Omega(B)$, then Theorem 2.28 (Derivations Version) also holds, by applying Ω^{-1} to Theorem 2.29 (Consecutive Derivations Version). This thus shows that the two theorems are equivalent. □

Let us now prove Theorem 2.29 (Consecutive Derivations Version). In order to do so, we will prove a lemma which implies Theorem 2.29 (Consecutive Derivations Version).

Lemma 2.30. *Let $A, B \subseteq \mathbb{N}$ with $0 \in A \cap B$ and such that every derivation T of A and B yields $g(B^T) = g(B) = g$. Suppose further that the set $\{f(B^T)\}$ is bounded for every derivation T. Then, for any finite subset F of A, there exists an integer $y \in \mathbb{N}^*$ and a derivation T of A and B, such that*

$$(F + y) \cup (F + y + g) \subseteq A^T. \tag{11}$$

Let $S = \{a_0, a_0+1, \ldots, a_0+g-1\}$ denote the set of g consecutive integers in A. We apply Lemma 2.30 for $F = S$, (11) gives us that there is a derivation T such that A^T contains $\{a_0+y, a_0+y+1, \ldots, a_0+y+2g-1\}$, we now have a set of $2g$ consecutive integers. We may apply to A^T and B^T again

Lemma 2.30 to get a set with $3g$ consecutive integers for a derivation T_1 of A^T and B^T. We can apply this arbitrarily many times and use Corollary 2.23 with $m_i = ig$ which directly implies Theorem 2.29 (Consecutive Derivations Version).

We are now ready to prove Lemma 2.30 in order to complete the proof of Theorem 2.16 (Subset Version).

Proof. Let us first recall that for any derivation T of A and B, the set $\{f(B^T)\}$ is bounded for any derivation T_1 of A^T and B^T. Let us write $f = \max_T f(B^T)$. Let $\mathcal{T}$ be the set of derivations T of A and B such that $f(B^T) = f$. We see that $\mathcal{T}$ is closed under the derivation operation.

We take $T_1 \in \mathcal{T}$ such that the number of residue classes modulo f is minimal in B^{T_1}. We denote the residue classes by $p_1 \dots, p_r$. Using Lemma 2.18, we see that for any derivation T of A^{T_1} and B^{T_1}, we have that $B^{T_1 T}$ lies among exactly those residue classes. We thus constructed a new set Σ of derivations T of A and B that minimizes the residue classes of B^T. Suppose $(A^T, B^T) \in \Sigma$ for some derivation T, we have the following properties:

(i) $g(B^T) = g(B) = g$

(ii) $f(B^T) = f$

(iii) B^T lies exactly among the residue classes $p_1, \dots, p_r$

(iv) Σ is closed under derivation.

Combining (i) and (iii) gives us that there exists non-negative integers $n_1, \dots n_r$ such that $g \equiv \sum_{i=1}^{r} n_i p_i \pmod f$. Equivalently for $s = \sum_{i=1}^{r} n_i$, we can write $g \equiv \sum_{i=1}^{s} \sigma_i \pmod f$, where $\sigma_i \in \{p_1, \dots, p_r\}$.

Let $T \in \Sigma$. Since $F \subseteq A$, $F \subseteq A^T$. By applying Lemma 2.19 to A^T and B^T, there exists a derivation T_1 such that $F + B^{TT_1} \subseteq A^{TT_1}$ and such that $(A^{TT_1}, B^{TT_1}) \in \Sigma$. We then use (iii) to have an $x_1 \equiv \sigma_1 \pmod f$ such that $F + x_1 \subseteq A^{TT_1}$. We then denote $F_1 = F + x_1$ and apply the same procedure with a derivation T_2 of A^{TT_1} and B^{TT_1}. We iterate this process s times such that we have a derivation $T^* = TT_1T_2 \dots T_s$ and $x_1, \dots x_s$ with $F + (x_1 + \dots + x_s) \subseteq A^{T^*}$ and $x_i = \sigma_i$ for all $i \in [s]$.

However we know that $\sum_{i=1}^{s} x_i = g + nf$ for some $n \in \mathbb{N}$ and that $F \subseteq A^{T^*}$ which means $F \cup (F + g + nf) \subseteq A^{T^*}$. If $n = 0$, we set $y = 0$ and we are done. If $n \neq 0$, by Lemma 2.19 we can find a derivation T' such that $\{F \cup (F + g + nf)\} + B^{T'} \subseteq A^{T'}$. Using (iv), we know $(A^{T'}, B^{T'}) \in \Sigma$, hence by (ii) $B^{T'}$ contains two elements which differ by f, suppose these are x and $x + f$. We thus have the two following inclusions

$$(F + x) \cup (F + x + g + nf) \subseteq A^{T'}, \tag{12}$$

and

$$(F + x + f) \cup (F + x + g + (n + 1)f) \subseteq A^{T'}. \tag{13}$$

If $n < 0$, we set $F' = F + x$ and if $n > 0$, we set $F' = F + x + f$. When $n < 0$, we combine (12) and (13) to get

$$F' \cup (F' + g + (n + 1)f) \subseteq A^{T'},$$

and when $n > 0$, we combine them to get

$$F' \cup (F' + g + (n - 1)f) \subseteq A^{T'}.$$

In either case, the coefficient of f is one closer to 0. We thus repeat this procedure n times, until arriving at a derivation T'' such that $(A^{T''}, B^{T''}) \in \Sigma$, a set F'' and a $y \in \mathbb{N}$, such that $F'' = F + y$ and

$$F'' \cup (F'' + g) \subseteq A^{T''}.$$

□

As mentioned, the proof of Lemma 2.30 implies Theorem 2.29 (Consecutive Derivations Version) and Theorem 2.28 (Derivations Version), which concludes the proof Theorem 2.16 (Subset Version).

2.6 Proof of Degenerate Version *bis* and Non-degenerate Version *bis*

So far, we always assumed that $0 \in A \cap B$. It was a particularly important invariant of τ-transforms. We may rewrite Theorem 2.16 (Subset Version) without the assumption $0 \in A \cap B$ in the following form.

Theorem 2.31 (Subset Version *bis*)**.** *Let $A, B \subseteq \mathbb{N}$. If $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, there must exist for every $c \in A+B$, a subset $C_c \subseteq A+B$ and an integer $g_c \in \mathbb{N}^*$ such that $c \in C_c$, $C_c^{(g_c)} \sim C_c$ and $\underline{d}(C_c) \geq \underline{d}(A) + \underline{d}(B) - 1/g_c$.*

Proof. Let $c \in A+B$. We can write $c = a+b$, for $a \in A$ and $b \in B$. We denote $A' = A - a$ and $B' = B - b$. Hence $0 \in A' \cap B'$. By Lemma 2.4 we have

$$\underline{d}(A'+B') = \underline{d}(A+B) \quad \text{ and } \quad \underline{d}(A') + \underline{d}(B') = \underline{d}(A) + \underline{d}(B).$$

By applying Theorem 2.16 (Subset Version) to A' and B', we get a subset $C' \subseteq A' + B'$, which contains 0, and we get an integer $g = g_c$ such that $C' \sim C'^{(g_c)}$ and $\underline{d}(C') \geq \underline{d}(A') + \underline{d}(B') - 1/g_c$. We conclude the proof by setting $C_c = C' + c$ and applying Lemma 2.4. ☐

We can now complete the proof of Theorem 2.15 (Non-degenerate Version *bis*). Namely, *if $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, there exists an integer g such that $A+B \sim (A+B)^{(g)}$.* First, observe that the set of all g_c is bounded as, if g_c can be arbitrarily large, the inequality $\underline{d}(C') \geq \underline{d}(A') + \underline{d}(B') - 1/g_c$ becomes $\underline{d}(A+B) \geq \underline{d}(C') \geq \underline{d}(A') + \underline{d}(B') = \underline{d}(A) + \underline{d}(B)$, contradicting the assumption.

Let g denote the least common multiple of all g_c. Using Theorem 2.31 (Subset Version *bis*), for every $c \in A+B$ and every $t \in \mathbb{N}$ large enough, we have $c + tg_c \in A+B$. Since g_c divides g, $A+B$ contains for large enough t' every integer of the form $c + t'g$. Take a finite set of representatives of the congruences classes of $(A+B)^{(g)}$, chosen from $(A+B) \cap (A+B)^{(g)}$. Having all integers of the form $c + t'g$ is equivalent to saying $A+B \sim (A+B)^{(g)}$, concluding the proof of Theorem 2.15 (Non-degenerate Version *bis*).

To prove Theorem 2.14 (Degenerate Version *bis*), namely that *if A and B are degenerate modulo some $g \in \mathbb{N}$, then there exists a minimal divisor g' of g such that $(A+B)^{(g')} = A+B$ and $\underline{d}(A^{(g')} + B^{(g')}) \geq \underline{d}(A^{(g')}) + \underline{d}(B^{(g')}) - 1/g'$*, we generalize Theorem 2.31 (Subset Version *bis*) for finite subsets.

Theorem 2.32 (Subset Version *ter*)**.** *Let $A, B \subseteq \mathbb{N}$ with $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$. For every finite subsets $\{c_1, \ldots, c_n\}$ of $A+B$, there exists a subset $C \subseteq A+B$ and an integer g such that $\{c_1, \ldots, c_n\} \subseteq C$, $C^{(g)} \sim C$, and $\underline{d}(C) \geq \underline{d}(A) + \underline{d}(B) - 1/g$.*

Proof. We proceed by induction on n. The case $n = 1$ is solved by Theorem 2.31 (Subset Version *bis*). Thus suppose $n > 1$ and the result is true for $\{c_1, \ldots, c_{n-1}\}$. Then there exists $C_1 \subseteq A+B$ and $l \in \mathbb{N}$, such that

$$\{c_1, \ldots, c_{n-1}\} \subseteq C_1 \quad , \quad C_1 \sim C_1^{(l)} \quad \text{and } \underline{d}(C_1) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{l}.$$

By Theorem 2.31 (Subset Version *bis*), there exists a subset $C_2 \subseteq A+B$ and an integer m such that

$$c_n \in C_2, \quad C_2 \sim C_2^{(m)}, \quad \text{ and } \quad \underline{d}(C_2) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{m}.$$

Let $C = C_1 \cup C_2$. Clearly, $\{c_1, \ldots c_n\} \subseteq C \subseteq A+B$. We then separate our proof in three cases.

First, if $C_1 \subseteq C_2^{(m)}$, we take $g = m$ and see that $C^{(g)} = (C_1 \cup C_2)^{(m)} = C_2^{(m)} \sim C_2 \subseteq C$. Since $C \subseteq C^{(g)}$, it follows that $C^{(g)} \sim C$ and $\underline{d}(C) \geq \underline{d}(C_2) \geq \underline{d}(A) + \underline{d}(B) - 1/g$.

The second case is when $C_2 \subseteq C_1^{(l)}$, here we take $g = l$ and proceed in the same manner.

The last case is when $C_1 \not\subseteq C_2^{(m)}$ and $C_2 \not\subseteq C_1^{(l)}$. Take $g = \operatorname{lcm}(l, m)$. We have

$$C \subseteq C^{(g)} = (C_1 \cup C_2)^{(g)} \subseteq (C_1^{(l)} \cup C_2^{(m)}) \sim (C_1 \cup C_2) = C.$$

We thus showed $C^{(g)} \sim C$. To prove the inequality, we observe that

(i) $C \sim (C_1^{(l)} \cup C_2^{(m)})$;

(ii) $\underline{d}(C_1^{(l)}) \geq \underline{d}(A) + \underline{d}(B) - 1/l$ and $\underline{d}(C_2^{(m)}) \geq \underline{d}(A) + \underline{d}(B) - 1/m$;

(iii) $C_1^{(l)} \cap C_2^{(m)}$ is a proper subset of $C_1^{(l)}$ and $C_2^{(m)}$.

Using (i), we reduce our case to proving that $\underline{d}(C_1^{(l)} \cup C_2^{(m)}) \geq \underline{d}(A) + \underline{d}(B) - 1/g$.

We write the residue class of $C_1^{(l)}$ and $C_2^{(m)}$ as $l_1 \dots, l_r$ and $m_1, \dots, m_s$ respectively. Using (iii), we have that $1 \leq r \leq l-1$ and $1 \leq s \leq m-1$. We also see that $\underline{d}(C_1^{(l)}) = r/l$ and $\underline{d}(C_2^{(m)}) = s/m$.

Using that $\underline{d}(C_1^{(l)} \cup C_2^{(m)}) = \underline{d}(C_1^{(l)}) + \underline{d}(C_2^{(m)}) - \underline{d}(C_1^{(l)} \cap C_2^{(m)})$, we know that if $\underline{d}(C_1^{(l)} \cap C_2^{(m)}) = 0$, then, by (ii),

$$\underline{d}(C_1^{(l)} \cup C_2^{(m)}) \geq \underline{d}(A) + \underline{d}(B) + \max(\frac{s}{m} - \frac{1}{l}, \frac{r}{l} - \frac{1}{m}) \geq \underline{d}(A) + \underline{d}(B),$$

which is better than needed. We may thus suppose that $\underline{d}(C_1^{(l)} \cap C_2^{(m)}) > 0$.

Let $d = \gcd(l, m)$. We may write $l = l_1 d$ and $m = m_1 d$, hence $g = l_1 m = l m_1$. We now partition $C_1^{(l)}$ and $C_2^{(m)}$ into d sets according to their residue class mod d. We note $C_{1,u}^{(l)}$ and $C_{2,v}^{(m)}$ their residue classes modulo d for $0 \leq u, v \leq d-1$. Let r_u be the number of residue classes (mod l) of $C_1^{(l)}$ in $C_{1,u}^{(l)}$ and similarly s_v the number of residue classes (mod m) of $C_2^{(m)}$ in $C_{2,v}^{(m)}$. Naturally, $0 \leq r_u \leq l_1$ and $0 \leq s_v \leq m_1$, for every $0 \leq u, v \leq d-1$. For an $u \in \{0, 1, \dots, d-1\}$, if $r_u = l_1$, we say that $C_{1,u}^{(l)}$ is *full*, and similarly for a $v \in \{0, 1, \dots, d-1\}$ and $C_{2,v}^{(m)}$ such that $s_v = m_1$.

We observe that $C_{1,u}^{(l)} \cap C_{2,v}^{(m)}$ must be empty whenever $u \neq v$ as they lie in different congruent classes, and additionally $\underline{d}(C_{1,u}^{(l)} \cap C_{2,u}^{(m)}) = r_u s_u / g$. This gives us

$$\underline{d}(C_1^{(l)} \cap C_2^{(m)}) = \sum_{u=0}^{d-1} \frac{r_u s_u}{g}, \tag{14}$$

and thus we can rewrite $\underline{d}(C_1^{(l)} \cup C_2^{(m)})$ as

$$\underline{d}(C_1^{(l)}) + \underline{d}(C_2^{(m)}) - \sum_{u=0}^{d-1} \frac{r_u s_u}{g} = \underline{d}(C_1^{(l)}) + \sum_{u=0}^{d-1} \frac{s_u l_1}{g} - \sum_{u=0}^{d-1} \frac{r_u s_u}{g} = \underline{d}(C_1^{(l)}) + \frac{1}{g} \sum_{u=0}^{d-1} s_u (l_1 - r_u). \tag{15}$$

By symmetry we also have $\underline{d}(C_1^{(l)} \cup C_2^{(m)}) = \underline{d}(C_2^{(m)}) + \frac{1}{g} \sum_{u=0}^{d-1} r_u (m_1 - s_u)$.

We reduce the proof to the case when for every $u \in \{0, 1, \dots, d-1\}$ we do not have that both $C_{1,u}^{(l)}$ and $C_{2,v}^{(m)}$ are full.

Indeed, suppose that there is $t > 0$ full congruence classes modulo d. We create the sets C_1' and C_2' by removing from the original sets the t full sets they have in common. We observe that

$$\underline{d}(C_1^{(l)}) - \underline{d}(C_1') = \underline{d}(C_2^{(m)}) - \underline{d}(C_2') = t/d.$$

Furthermore, C_1' and C_2' satisfy conditions (ii) and (iii) when replacing $\underline{d}(A) + \underline{d}(B)$ by $\underline{d}(A) + \underline{d}(B) - t/d$ and has no common full congruence class. So, proving that $\underline{d}(C_1^{(l)} \cup C_2^{(m)}) \geq \underline{d}(A) + \underline{d}(B) - 1/g$ is equivalent to showing that $\underline{d}(C_1' \cup C_2') \geq \underline{d}(A) + \underline{d}(B) - t/d - 1/g$, only this time there is no pair of full congruence classes. We will thus, without loss of generality, prove it when for every $u \in \{0, 1, \dots, d-1\}$, either $C_{1,u}^{(l)}$ or $C_{2,u}^{(m)}$ is not full.

As $\underline{d}(C_1^{(l)} \cup C_2^{(m)}) = \sum_{u=0}^{d-1} r_u s_u / g > 0$, there exists $u_0 \in \{0, 1, \dots, d-1\}$ such that $r_{u_0} s_{u_0} \geq 1$, and at least one of $C_{1,u_0}^{(l)}, C_{2,u_0}^{(m)}$ is not full.

If $C_{1,u_0}^{(l)}$ is full, that is $r_{u_0} = l_1$, by (15) and (ii),

$$\underline{d}(C_1^{(l)} \cup C_2^{(m)}) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{m} + \frac{1}{g} r_{u_0} (m_1 - s_{u_0}) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{m} + \frac{l_1}{g} = \underline{d}(A) + \underline{d}(B),$$

which is stronger than needed. By symmetry, we also solved the case when $C^{(m)}_{2,u_0}$ is full.

If neither $C^{(l)}_{1,u_0}$ nor $C^{(m)}_{2,u_0}$ are full, we have $1 \le r_{u_0} < l_1$ and $1 \le s_{u_0} < m_1$. Once more, (15) gives us

$$\underline{\mathrm{d}}(C_1^{(l)} \cup C_2^{(m)}) \ge \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B) - \frac{1}{g} + \max\left(\frac{1}{g} - \frac{1}{l} + \frac{s_{u_0}}{g}(l_1 - r_{u_0}), \frac{1}{g} - \frac{1}{m} + \frac{r_{u_0}}{g}(m_1 - s_{u_0})\right).$$

To conclude the proof, we just need to show that one of the terms in the max is bigger or equal to 0. Alternatively, we can show that the sum, let us call it S, is itself bigger than 0, which is stronger. We multiply S by g to get

$$\begin{aligned} g \cdot S &\ge 1 - \frac{g}{l} + s_{u_0}(l_1 - r_{u_0}) + 1 - \frac{g}{m} + r_{u_0}(m_1 - s_{u_0}) \\ &= s_{u_0}(l_1 - r_{u_0}) + r_{u_0}(m_1 - s_{u_0}) - l_1 - m_1 + 2 \\ &= l_1(s_{u_0} - 1) + m_1(r_{u_0} - 1) - 2r_{u_0}s_{u_0} + 2 \\ &\ge (r_{u_0} + 1)(s_{u_0} - 1) + (s_{u_0} + 1)(r_{u_0} - 1) - 2r_{u_0}s_{u_0} + 2 = 0, \end{aligned}$$

and so $S \ge 0$, which shows that $\underline{\mathrm{d}}(C_1^{(l)} \cup C_2^{(m)}) \ge \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B) - 1/g$. □

We end the proof of Theorem 2.2 (Kneser's Theorem) by showing that Theorem 2.32 (Subset Version *ter*) implies Theorem 2.14 (Degenerate Version *bis*). Let $A, B \subseteq \mathbb{N}$, and suppose that the pair is degenerate modulo g. Then $A + B = (A + B)^{(g)}$, and let g' be the smallest natural number with the property that $A+B = (A+B)^{(g')}$. By Lemma 2.13, g' divides g and $(A+B)^{(g')} = A^{(g')} + B^{(g')}$. We need to show

$$\underline{\mathrm{d}}(A^{(g')} + B^{(g')}) \ge \underline{\mathrm{d}}(A^{(g')}) + \underline{\mathrm{d}}(B^{(g')}) - \frac{1}{g'},$$

and we may thus assume that $\underline{\mathrm{d}}(A^{(g')} + B^{(g')}) < \underline{\mathrm{d}}(A^{(g')}) + \underline{\mathrm{d}}(B^{(g')})$.

Let $c_1, \ldots, c_n$ be the set of representatives of the different congruence classes (mod g') in $(A+B)^{(g')}$. We apply Theorem 2.32 (Subset Version *ter*) to $A^{(g')}$ and $B^{(g')}$. This gives us a subset $C \subseteq (A+B)^{(g')}$ such that $c_1, \ldots, c_n \in C$ and hence $C^{(g')} = (A + B)^{(g')}$. It also gives us an integer $g'' \in \mathbb{N}^*$ such that $C^{(g'')} \sim C$ and $\underline{\mathrm{d}}(C) \ge \underline{\mathrm{d}}(A^{(g')}) + \underline{\mathrm{d}}(B^{(g')}) - 1/g''$.

Let $d = \gcd(g', g'')$. By Lemma 2.13, we have

$$(A + B)^{(d)} = \left((A + B)^{(g')}\right)^{(g'')} = \left(C^{(g')}\right)^{(g'')} = \left(C^{(g'')}\right)^{(g')} = C^{(g')} = (A + B)^{(g')}.$$

By the minimal property of g', we have $d = g'$ and hence $g'' \ge g'$. This gives us

$$\underline{\mathrm{d}}((A + B)^{(g')}) = \underline{\mathrm{d}}(C^{(g')}) \ge \underline{\mathrm{d}}(C) \ge \underline{\mathrm{d}}(A^{(g')}) + \underline{\mathrm{d}}(B^{(g')}) - 1/g'' \ge \underline{\mathrm{d}}(A^{(g')}) + \underline{\mathrm{d}}(B^{(g')}) - 1/g',$$

finishing up the proof of Theorem 2.14 (Degenerate Version *bis*) and thus of Theorem 2.2 (Kneser's Theorem). □

3 Mann's theorem in higher dimensions

3.1 One-dimensional case

Where Kneser's Theorem deals with *asymptotic* density, the tools were first developed for a much stricter definition of density, one where if $A \subseteq \mathbb{N}$ with $1 \notin A$, A would have density 0. This is called the Schnirelmann density and puts an emphasis on the beginning of subsets through the use of the infimum.

Definition 3.1 (Schnirelmann's density)**.** Let $A \subseteq \mathbb{N}$. We define the Schnirelmann density of A, $\sigma(A)$, as

$$\sigma(A) = \inf_{n \in \mathbb{N}^*} \frac{|A \cap [1, n]|}{n}. \tag{16}$$

We see by definition that $\sigma(A) \leq \underline{d}(A)$, for any set $A \subseteq \mathbb{N}$. In some sense, having a definition of density yielding smaller values may help having the density of $A+B$ to be bigger than the sum of the densities of A and B. Indeed, if we were to "remove" $\varepsilon > 0$ from the definition of density, we would remove it twice when summing the density, while only removing it once from the density of $A + B$.

Additionally, the use of inf instead of liminf, prevents A and B from being eventually periodic, it needs to be periodic from the ground up to have the desired density. If A and B are periodic, say modulo some prime p, Cauchy-Davenport would tell us that $\sigma(A + B) \geq \sigma(A) + \sigma(B) - 1/p$. If we force $0 \in A \cap B$, as we do not count 0 in the Schnirelmann's density, the density would be calculated at $n = p - 1$.

So, if we had $\sigma(A+B) = k/p$, $\sigma(A) = k_1/p$, and $\sigma(B) = k_2/p$ with $k = k_1 + k_2 - 1$, the density would be changed to $\sigma(A + B) = (k - 1)/(p - 1)$, $\sigma(A) = (k_1 - 1)/(p - 1)$, and $\sigma(B) = (k_2 - 1)/p$.

And so we have $\sigma(A + B) = \sigma(A) + \sigma(B)$. We thus may be led to believe that the periodic case in Kneser's Theorem, will not happen with the Schnirelmann density. In 1930, L. G. Schnirelmann made a first step in this direction.

Theorem 3.2 (Schnirelmann's Theorem A)**.** *Let $A, B \subseteq \mathbb{N}$. If $0 \in A \cap B$, we have $\sigma(A + B) \geq \sigma(A) + \sigma(B) - \sigma(A)\sigma(B)$.*

Theorem 3.3 (Schnirelmann's Theorem B)**.** *Let $A, B \subseteq \mathbb{N}$. If $1 \in A$ and $0 \in B$ and $\sigma(A)+\sigma(B) \geq 1$, we have $A + B = \mathbb{N}$.*

The proofs of Theorem 3.2 (Schnirelmann's Theorem A) and Theorem 3.3 (Schnirelmann's Theorem B) are quite short and can be found here, [Sch33]. While not being the *full* result, these were enough to tackle the work sought by L. G. Schnirelmann. In particular, in the same paper, [Sch33], he also showed that any integer $n > 1$ can be written as the sum of at most C primes, where C is a finite constant. Proving that $C = 3$ would solve Goldbach's conjecture, [Gol42].

In 1942, H. B. Mann managed to remove the $\sigma(A)\sigma(B)$ factor from Theorem 3.2 (Schnirelmann's Theorem A), yielding the following theorem.

Theorem 3.4 (Mann's Theorem)**.** *Let $A, B \subseteq \mathbb{N}$. If $0 \in A \cap B$, we have*

$$\sigma(A + B) \geq \min(1, \sigma(A) + \sigma(B)).$$

The original proof is available here [Man42], and was later simplified by E.Artin and P. Scherk in [AS43].

This combines both Theorem 3.2 (Schnirelmann's Theorem A) and Theorem 3.3 (Schnirelmann's Theorem B) while strengthening the result to be a proper equivalent to Kneser's Theorem for the Schnirelmann density.

It is important to note that Theorem 3.4 (Mann's Theorem) was proved before Theorem 2.2 (Kneser's Theorem). So, properly speaking, Kneser's Theorem is the lower asymptotic density version of Mann's Theorem and not the other way around.

3.2 Exploration of the two-dimensional case

Naturally, we wonder if this theorem extends for $A, B \subseteq \mathbb{N}^2$, for every $k \in \mathbb{N}^*$ and $A, B \subseteq \mathbb{N}^k$, or even for $A, B \subseteq \mathbb{N}^{<\infty}$ (all finite sequences of natural numbers). To answer this question, we first need to find a suitable definition of σ in $\mathbb{N}^2$. One possibility is to look at the square boxes, which yields the following definition.

Definition 3.5. Let $A \subseteq \mathbb{N}^2$. We define the *square density* of A as

$$\sigma_\square(A) = \inf_{n \in \mathbb{N}^*} \frac{|A \cap [1,n]^2|}{n^2}. \tag{17}$$

Unfortunately, we do not have Mann's Theorem with $\sigma_\square$. There exists $A, B \subseteq \mathbb{N}^2$ with $(0,0) \in A \cap B$ such that

$$\sigma_\square(A+B) < \min(1, \sigma_\square(A) + \sigma_\square(B)).$$

Example 3.6. Take $A = B = \{(0,0)\} \cup \{(x,y) : x \geq 1, y \geq 1\} \backslash \{(1,2)\}$. We see that

$$\sigma_\square(A) = \sigma_\square(B) = \frac{|A \cap [1,2]^2|}{2^2} = \frac{3}{4}.$$

We note that $(1,2) \notin A+B$ and thus we have $\sigma_\square(A+B) = 3/4$ which shows

$$\sigma_\square(A+B) < \min(1, \sigma_\square(A) + \sigma_\square(B)).$$

We might thus define σ on rectangles instead of squares only, as we used the asymmetry of A and B.

Additionally we note that, if we set $A = B = \{(0,0)\} \cup \{(x,y) : x \geq 1, y \geq 1\} \backslash \{1,2\}$, we have $A+B = A$ and so for Mann's theorem to apply we must either have $\sigma(A) = 0$ or $\sigma(A) = 1$. The latter does not make much sense as in Definition 3.1 (Schnirelmann's density) the beginning matters a lot through the inf.

We thus want to find a definition of σ where $\sigma(\{(0,0)\} \cup \{(x,y) : x \geq 1, y \geq 1\} \backslash \{1,2\}) = 0$. This can be achieved through taking rectangles, starting from 0.

Definition 3.7. Let $A \subseteq \mathbb{N}^2$. We define the *rectangular density* of A as

$$\sigma_*(A) = \inf_{n,m \in \mathbb{N}^*} \frac{|A \cap [0,n] \times [0,m]|}{(n+1)(m+1)}. \tag{18}$$

Here σ_* yields smaller values to subsets of $\mathbb{N}^2$ but still fails to have a Mann-like theorem. We need to be careful as in Theorem 3.4 (Mann's Theorem), if $1 \notin A$ we had $\sigma(A) = 0$, which is not the case anymore. So we might want to force $(1,1) \in A \cap B$.

Unfortunately, there exists $A, B \subseteq \mathbb{N}^2$ with $\{(0,0), (1,1)\} \in A \cap B$ such that

$$\sigma_*(A+B) < \min(1, \sigma_*(A) + \sigma_*(B)).$$

Example 3.8. Take $A = B = \mathbb{N}^2 \backslash \{(0,1), (1,0)\}$. We have $\sigma_*(A) = \frac{1}{2}$ and $A+B = A$. So we indeed have $\sigma_*(A+B) < \min(1, \sigma_*(A) + \sigma_*(B))$.

Even if we take the full square $[0,1]^2$, we cannot prove Mann's theorem for $\mathbb{N}^2$. As there exists $A, B \subseteq \mathbb{N}^2$ with $[0,1]^2 \in A \cap B$ such that

$$\sigma_*(A+B) < \min(1, \sigma_*(A) + \sigma_*(B)).$$

Example 3.9. Take $A = [0,1]^2 \cup \left(\mathbb{N}^2 \backslash [0,3]^2\right)$ and $B = \mathbb{N}^2 \backslash [2,3]^2$. We have $\sigma_*(A) = 1/4$ and $\sigma_*(B) = 3/4$, so we must show $\sigma_*(A+B) < 1$.

We see that $(3,3) \notin A+B$ which means

$$\sigma_*(A+B) \leq \frac{|(A+B)\cap[0,3]^2|}{4^2} \leq \frac{15}{16} < 1.$$

Indeed, to have $(3,3) \in A+B$ we would need $a \in A$ such that $a \in [0,1]^2$ but this implies that there is a $b \in B$ such that $b \in [2,3]^2$ which is impossible by definition.

This example is very insightful as we can scale it up to accommodate any finite constraints.

Lemma 3.10. *Let $X \subseteq \mathbb{N}^2$ finite. There exists $A, B \subseteq \mathbb{N}$ with $X \subseteq A \cap B$ such that*

$$\sigma_*(A+B) < \min(1, \sigma_*(A)+\sigma_*(B)).$$

Proof. As X is finite, there exists $n \in \mathbb{N}$ odd such that $X \subseteq [0,n]^2$. Now we define $A = [0,n]^2 \cup \mathbb{N}^2 \backslash [0,2n+1]^2$ and $B = \mathbb{N}^2 \backslash [n+1, 2n+1]^2$. We can compute that A and B's worst rectangles are when we look at $[0,2n+1]^2$. We thus have $\sigma_*(A) = 1/4$ and $\sigma_*(B) = 3/4$. We conclude by showing that $\sigma_*(A+B) < 1$ which is easy to see as $(2n+1, 2n+1) \notin A+B$. ☐

Maybe we should not look at finite constraints but as the fact that in the 1-dimensional version, we have a 0-dimensional constraint ($0 \in A \cap B$), so in the 2-dimensional version, it is reasonable to look at 1-dimensional constraints. Note that we cannot hope to ask for both axes as if we have $(\{0\}\times\mathbb{N}) \cup (\mathbb{N}\times\{0\}) \subseteq A \cap B$ then $A+B = \mathbb{N}^2$ making the theorem trivial. Unfortunately, asking for one axis does not solve our problem.

We can find $A, B \subseteq \mathbb{N}^2$ with $\{0\}\times\mathbb{N} \subseteq A \cap B$ such that

$$\sigma_*(A+B) < \min(1, \sigma_*(A)+\sigma_*(B)).$$

Example 3.11. We only need to slightly modify Example 3.9 to make it work. Take, as in Example 3.9 $A = [0,1]^2 \cup \left(\mathbb{N}^2\backslash[0,3]^2\right)$ and $B = \mathbb{N}^2\backslash[2,3]^2$. Unfortunately, $A \not\subseteq \{0\}\times\mathbb{N}$, so we define $A' = A \cup \{(0,2),(0,3)\}$ and to ensure $(3,3) \notin A+B$, we define $B' = B\backslash\{(3,0),(3,1)\}$. We still have $\sigma_*(A')+\sigma_*(B') = 1$ as $\sigma_*(A') = 3/8$ and $\sigma_*(B') = 5/8$. Additionally our transformation satisfies $(3,3) \notin A'+B'$ so we indeed have $\sigma_*(A'+B') < 1$.

We can even extend this range arbitrarily to scale up Example 3.11.

Lemma 3.12. *Let $n \in \mathbb{N}^*$. There exists $A, B \subseteq \mathbb{N}^2$ with $\{0,1,\ldots,n\}\times\mathbb{N} \subseteq A \cap B$ such that*

$$\sigma_*(A+B) < \min(1, \sigma_*(A)+\sigma_*(B)).$$

Proof. We combine the examples in Lemma 3.10 and Example 3.11. We define $A = B = \mathbb{N}^2\backslash([n+1, 2n+1]\times[0,2n+1])$. It is quite clear that $\sigma_*(A) = \sigma_*(B) = 1/2$, so we just need to show that $\sigma_*(A+B) < 1$. This is easily seen by observing that $(2n+1, 2n+1) \notin A+B$. ☐

Corollary 3.13. *Let $X \subseteq \mathbb{N}$ finite. There exists $A, B \subseteq \mathbb{N}^2$ with $X \times \mathbb{N} \subseteq A \cap B$ such that*

$$\sigma_*(A+B) < \min(1, \sigma_*(A)+\sigma_*(B)).$$

Proof. We note that for every finite $X \subseteq \mathbb{N}$, there exists an $n \in \mathbb{N}^*$ such that $X \subseteq \{0,1,\ldots,n\}$ and apply Lemma 3.12. ☐

If the constraints are not the way to find a multidimensional version of Mann's Theorem, maybe we should continue to explore ways to define the density.

A much more restrictive version of density is to look at fibers of A and B. Let $A \subseteq \mathbb{N}^2$, we define $A_i = A \cap (\{i\}\times\mathbb{N})$. From this we can define the density over fibers.

Definition 3.14. Let $A \subseteq \mathbb{N}^2$. The *fiber density* of A is defined as

$$\sigma_\|(A) = \inf_{i\in\mathbb{N}^*} \sigma(A_i) = \inf_{i\in\mathbb{N}^*} \inf_{n\in\mathbb{N}^*} \frac{|A_i \cap (\{i\}\times[1,n])|}{n}.$$

Remark 3.15. It would be fair to dislike this definition as it is very much not symmetrical, one could make it symmetrical by defining σ as the max between horizontal and vertical fibers. However this is equivalent to only looking at vertical fibers by noting that if in A we have vertical fibers and in B horizontal ones, $A + B = \mathbb{N}^2$ making the theorem trivial.

We can then restrict ourselves to try and prove Mann's theorem with $\sigma_{\|}$ and it turns out that for once the theorem is true!

Proposition 3.16. *Let $A, B \subseteq \mathbb{N}^2$ with $(0,0) \in A \cap B$. Then we have*

$$\sigma_{\|}(A + B) \geq \min(1, \sigma_{\|}(A) + \sigma_{\|}(B)).$$

Proof. First, note that for $i \geq 1$, we have $A_1 + B_{i-1} \subseteq (A + B)_i$ and thus by Theorem 3.4 (Mann's Theorem) we have $\sigma((A+B)_i) \geq \min(1, \sigma(A_1)+\sigma(B_{i-1}))$. Then we use the fact that $\sigma(A_1) \geq \sigma_{\|}(A)$ and $\sigma(B_{i-1}) \geq \sigma_{\|}(B)$ to show that for every $i \geq 1$ we have $\sigma((A+B)_i) \geq \min(1, \sigma(A_1)+\sigma(B_{i-1})) \geq \min(1, \sigma_{\|}(A) + \sigma_{\|}(B))$ which implies that

$$\sigma_{\|}(A + B) \geq \min(1, \sigma_{\|}(A) + \sigma_{\|}(B)).$$

□

Although, we did prove a *version* of Mann's Theorem in two dimensions, it is not an interesting one. Proposition 3.16 is actually weaker than Theorem 3.4 (Mann's Theorem), so it is still not a good two-dimensional version. We need to look further. Maybe instead of trying to find the *right* definition of density, we could instead look at a weighted version of Mann's Theorem.

Question 3.17. What is the smallest $c \in \mathbb{R}_{>1}$ such that for $A, B \subseteq \mathbb{N}^2$ with $(0,0) \in A \cap B$ we have

$$c \cdot \sigma_{\square}(A + B) \geq \min(c, \sigma_{\square}(A) + \sigma_{\square}(B))?$$

Answer. The answer is the *trivial* $c = 2$. Let us first show that $c \geq 2$. We will proceed very similarly to Example 3.6. Take $A = B = \{(0,0)\} \cup \{(x,y) : x \geq 0 \text{ and } y \geq 0\} \backslash \{(1,2),(2,1)\}$. We note that $A + B = A$ and that $\sigma_{\square}(A) = 1/2$. So we need to have $c \cdot 1/2 \geq \min(1, 1/2 + 1/2) = 1$. This shows that $c \geq 2$.

We are now left to show that $c \leq 2$. Without loss of generality, suppose that $\sigma_{\square}(A) \geq \sigma_{\square}(B)$ (otherwise switch A and B). As $(0,0) \in B$ we know that $A \subseteq A+B$ and thus that $\sigma_{\square}(A+B) \geq \sigma_{\square}(A)$.

We conclude by noting that

$$\min(c, \sigma_{\square}(A) + \sigma_{\square}(B)) \leq \min(c, 2\sigma_{\square}(A)) \leq \min(c, 2\sigma_{\square}(A + B)) \leq 2\sigma_{\square}(A + B),$$

where the last inequality is true when $c = 2$, so the minimal c must be less than or equal to 2. □

We just showed the c satisfying Question 3.17 is the trivial one, which is not particularly interesting either. Up to here, our counterexamples are "finite", in the sense that after some point we have everything. In what follows, we show that this is not a necessity for $\sigma_{\square}$.

There exists $A, B \subseteq \mathbb{N}^2$ with $(0,0) \in A \cap B$ such that for every $n \in \mathbb{N}$ we have $A + B \backslash [0,n]^2 \neq \emptyset$ and that

$$\sigma_{\square}(A + B) < \min(1, \sigma_{\square}(A) + \sigma_{\square}(B)).$$

Example 3.18. We define $A = B = \{(a,b) \in \mathbb{N}^2 : a \geq b\}$. We note that A and B have exactly half of each $[0,n]^2$ apart from the extra half of the diagonal. So we have

$$\sigma_{\square}(A) = \sigma_{\square}(B) = \inf_{n \in \mathbb{N}^*} \frac{A \cap [1,n]^2}{n^2} = \inf_{n \in \mathbb{N}^*} \frac{n^2 + n}{2n^2} = \frac{1}{2}.$$

Furthermore, we see that $A + B = \{(a_1 + a_2, b_1 + b_2) : a_1, a_2, b_1, b_2 \in \mathbb{N}, a_1 \geq b_1, a_2 \geq b_2\} = A$ as $a_1 + a_2 \geq b_1 + b_2$ so $A + B \subseteq A$ (and clearly $A \subseteq A + B$).

Whilst searching for an "infinite" example where Mann's Theorem would fail with σ_*, we realized that what we were probably looking for was not an infinite example but rather a slightly different definition of the rectangular density, where 0 is not considered while still considering rectangles, just like in Definition 3.1 (Schnirelmann's density).

3.3 Research in the area

Remark 3.19. In this subsection, to simplify the notation, we will consider $A + B = \{a, b, a + b : a \in A, b \in B\}$ regardless of whether $0 \in A$ and $0 \in B$.

There are still multiple ways to generalize the notion of density to higher dimensions. If we continue to consider the two-dimensional case and take rectangles without $(0, 0)$, we have some interesting results found by Luther Cheo in [Che51].

He mapped $\mathbb{N}^2$ to the first quadrant of the gaussian integers $Q = \{x + yi : x, y \in \mathbb{N}\} \backslash \{(0, 0)\}$. Let $A \subseteq Q$. We write $A(x + yi)$ as the number of $a + bi \in Q$ such that $a \leq x$ and $b \leq y$. We now define the density.

Definition 3.20 (Cheo density)**.** Let $A \subseteq Q$. We define the *Cheo density* of A as

$$\sigma_C(A) = \inf_{x+yi \in Q} \frac{A(x + yi)}{xy + x + y}.$$

Cheo then showed two things. First, an equivalent to Theorem 3.3 (Schnirelmann's Theorem B).

Theorem 3.21. *Let $A, B \subseteq Q$. If $\sigma_C(A) + \sigma_C(B) \geq 1$, then $\sigma_C(A + B) = 1$.*

Proof. We proceed like in [Che51]. Suppose that $\sigma_C(A + B) < 1$. We then have an element $c + c'i \notin A + B$. For any $a + a'i \in A$ we have that $c + c'i - (a + a'i) = c - a + (c' - a')i \notin B$. Then at least one element $x + yi$ such that $x \leq c$ and $y \leq c'$ is not in either A or B and thus $A(c + c'i) + B(c + c'i) \leq cc' + c + c' - 1$. This yields

$$\frac{A(c + c'i) + B(c + c'i)}{cc' + c + c'} < 1,$$

hence $\sigma_C(A) + \sigma_C(B) < 1$, a contradiction. □

Then, he showed a weaker version of Theorem 3.2 (Schnirelmann's Theorem A).

Theorem 3.22. *Let $A, B \subseteq Q$. If for every $k \in \mathbb{N}$ we have that $0 + ki \in B$, then $\sigma_C(A + B) \geq \sigma_0(A) + \sigma_C(B) - \sigma_0(A)\sigma_C(B)$, where*

$$\sigma_0(A) = \inf_{x \in \mathbb{N}^*} \frac{A(x + 0i)}{x}.$$

Proof. The proof is available in [Che51]. □

The conditions to have all $ki \in B$ is quite restrictive, especially if one wants to prove the generalization of Mann's Theorem with σ_C. And indeed, it turns out that Cheo found an example where the generalization breaks.

Example 3.23 (Cheo sets)**.** We define two sets $A, B \subseteq Q$, such that $x + yi \in A \cap B$ whenever $x > 20$ or $y > 1$. Here are the representations of A, B, and $A + B$, where $\bullet$ indicates that an element is in the set and $\cdot$ that it is not.

Set A

Im(z) = 1	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$
Im(z) = 0	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$
Re(z)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Set B

Im$(z) = 1$	•	·	·	•	•	·	·	·	•	·	·	·	•	·	·	·	•	·	·	·	•
Im$(z) = 0$	·	•	·	·	•	•	·	·	•	•	·	·	·	•	·	·	·	•	·	·	·
Re(z)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Set A+B

Im$(z) = 1$	•	•	•	•	•	•	•	·	•	•	•	·	•	•	•	·	•	•	•	·	•
Im$(z) = 0$	·	•	•	•	•	•	•	•	•	•	•	•	·	•	•	•	·	•	•	•	·
Re(z)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Note that $\sigma_C(A) = 1/2$, $\sigma_C(B) = 1/3$, and $\sigma_C(A+B) = 34/41$. This shows that

$$\sigma_C(A+B) = \frac{34}{41} < \frac{1}{2} + \frac{1}{3} = \sigma_C(A) + \sigma_C(B).$$

This means that if we want a Mann's Theorem in two dimensions we cannot take this definition of density. One other way to define density seen in the literature is to look at *fundamental* sets. Here, we directly consider directly the n-dimensional case. We define the non-zero lattice points $Q_n = \mathbb{N}^n \backslash \{\mathbf{0}\}$.

Definition 3.24. We call a finite, non-empty, subset $R \subseteq Q_n$ a *fundamental set*, if for any element $(r_1, \ldots, r_n) \in R$, we have that all $(x_1, \ldots, x_n) \in Q_n$ with $x_i \leq r_i, \forall\, i \in [n]$, are also in R.

We are now ready to define the generalized Schnirelmann density.

Definition 3.25. Let $A \subseteq Q_n$. We define the n-dimensional Schnirelmann density of A as

$$\sigma_n(A) = \inf_R \frac{|A \cap R|}{|R|},$$

where the inf ranges over all fundamental sets R of Q_n.

Using this generalization of the Schnirelmann density, Betty Kvarda showed generalizations of Theorem 3.2 (Schnirelmann's Theorem A) and Theorem 3.3 (Schnirelmann's Theorem B) in [Kva63].

Theorem 3.26. *Let $A, B \subseteq Q_n$. If $\sigma_n(A) + \sigma_n(B) \geq 1$, then we have $\sigma_n(A+B) = 1$.*

Proof. The proof is very similar to Theorem 3.22's. Suppose that $\sigma_n(A+B) < 1$. Then, there exists an element $x = (x_1, \ldots, x_n) \in Q_n$ such that $x \notin A+B$. Let R be the set of all elements $(y_1, \ldots, y_n) \in Q_n$ with $y_i \leq x_i$ for all $i \in [n]$.

Now, for any $y \in R$, either $y \in A$, or $y = x - b$ for some $b \in B$, or neither are true, but we know that not both $y \in A$ and $x - y \in B$ can be true for the same $y \in R$. In particular we know that neither $x \in A$ nor $x - x \in B$, and thus

$$|A \cap R| + |B \cap R| \leq |R| - 1.$$

Using that $\sigma_n(A) + \sigma_n(B) \leq (|A \cap R| + |B \cap R|)\,/|R|$, we get $\sigma_n(A) + \sigma_n(B) < 1$, a contradiction. □

Theorem 3.27. *Let $A, B \subseteq Q_n$. We have $\sigma_n(A+B) \geq \sigma_n(A) + \sigma_n(B) - \sigma_n(A)\sigma_n(B)$.*

Proof. We proceed like in [Kva63]. First, we denote $e_i \in Q_n$ the vector who is 1 at the i-th coordinate and 0 everywhere else. Note that, if there is an $i \in [n]$ such that $e_i \notin A$, then $\sigma_n(A) = 0$ and the result is trivial. We may thus assume that $e_i \in A$ for every $i \in [n]$.

Let $R \subseteq Q_n$ be any fundamental set. To prove Theorem 3.27, we need to show $|(A+B) \cap R|/|R| \geq \sigma_n(A) + \sigma_n(B) - \sigma_n(A)\sigma_n(B)$.

As $1 \geq \sigma_n(A) + \sigma_n(B) - \sigma_n(A)\sigma_n(B)$, we may assume that $|(A+B) \cap R| < |R|$, and thus $|A \cap R| < |R|$. Let $H = R \backslash A$. We show that there exists vectors $a^1, \ldots a^s \in A$ and sets $L_1, \ldots, L_s$ satisfying the following:

(i) The set $\{L_1, \dots, L_s\}$ partitions H.

(ii) The sets $L'_i = L_i - a^i$ are fundamental sets.

We start by ordering R, with $(x_1, \dots, x_n) > (x'_1, \dots, x'_n)$ if and only if there exists $k \in [s]$ such that $x_i = x'_i$ for all $i < k$ and $x_k > x'_k$. Now, let $h = (h_1, \dots, h_n) \in H$, we define the set $A_h = \{(a_1, \dots, a_n) : a_i \leq h_i \; \forall \, i \in [n]\}$. The sets A_h are non-empty as all e_i are in A. Since H is finite, A_h contains a maximal element, according to our ordering of R.

Let $a^1 \dots, a^s$ be the set of distinct maximal element amongst all A_h, and let L_i be the set of all $h \in H$ such that A_h has a^i as maximal element. Clearly (i) is satisfied.

To show (ii), set $x = (x_1, \dots, x_n) \in L_i$ and take any $y = (y_1, \dots, y_n)$ such that $x_j \geq y_j \geq a^i_j$ for every $j \in [n]$. If $y = a^i$, we know that $y \in L_i$. Suppose thus that $y \neq a^i$ and $y \in L_k$ for $k \neq i$. This implies $x_j \geq y_j \geq a^k_j$ and $a^k \geq a^i$, since a^k in maximal in A_y.

However using $x \in L_i$ and $x_j \geq a^k_j$, we know that $a^k \in A_x$ and that a^i is maximal in A_x which implies $a^i \geq a^k$, so $a^i = a^k$, a contradiction showing $y \in L_i$. Hence $L'_i = L_i - a^i$ is a fundamental set, showing (ii).

To conclude, let $b \in B \cap L'_i$. Then, $a^i + b \in A + B \cap L_i$, thus in $(A+B)\backslash A$. Therefore, we have $|((A+B)\backslash A) \cap R| \geq \sum_{i=1}^{s} |B \cap L'_i|$. This implies that $|(A+B) \cap R| \geq |A \cap R| + \sum_{i=1}^{s} |B \cap L'_i|$. Using $|L'_i| = |L_i|$, as $x \geq a^i$ for all $x \in L_i$, we get

$$\begin{aligned} |(A+B) \cap R| \geq |A \cap R| + \sum_{i=1}^{s} |B \cap L'_i| &\geq |A \cap R| + \sum_{i=1}^{s} \sigma_n(B) \cdot |L'_i| = |A \cap R| + \sigma_n(B) \sum_{i=1}^{s} |L_i| \\ &= |A \cap R| + \sigma_n(B) \cdot |H| = |A \cap R| + \sigma_n(B) \cdot (|R| - |A \cap R|) \\ &= (1 - \sigma_n(B)) \cdot |A \cap R| + \sigma_n(B) \cdot |R| \geq (1 - \sigma_n(B))\sigma_n(A)|R| + \sigma_n(B) \cdot |R|. \end{aligned}$$

And so,

$$\frac{|(A+B) \cap R|}{|R|} \geq \sigma_n(A) + \sigma_n(B) - \sigma_n(A)\sigma_n(B),$$

finishing the proof. □

Kvarda mentions at the very end of [Kva63] that the condition to have all $ki \in B$ in Cheo's paper [Che51] is necessary if we wish to use the same type of argument when partitioning H. Without going into the details, Kvarda mentions that this can be seen if we take the fundamental set $R = \{x + yi : x + yi \in Q, x \leq 4, y \leq 3\}$, and we let $A \subseteq Q$ such that $A \cap R = \{1, i, 3 + 3i\}$.

We can ask ourselves if something interesting may come out when taking an A such that $A \cap R = \{1, i, 3 + 3i\}$. In particular, let us have a look at the set $A = \{x + yi : x + yi \in Q, x + yi \in \{1, i, 3 + 3i\} \text{ or } x + yi \notin R\}$. Can we find a $B \subseteq Q$, such that $\sigma_C(A + B) < \min(\sigma_C(A) + \sigma_C(B), 1)$?

Short answer, no. With a small program that tests every possible set $B_{R'} = \{x + yi : x + yi \in Q, x + yi \in R' \text{ or } x + yi \notin R\}$, where R' ranges over all subsets $R' \subseteq R$, we can confirm that no such B exists. The program is available in Appendix B.

Three years after [Kva63], Allen R. Freedman managed to strengthen Theorem 3.27.

Theorem 3.28. *Let $A, B \subseteq Q_n$. We have $\sigma_n(A + B) \geq \sigma_n(B)/(1 - \sigma_n(A))$.*

Proof. The proof is available in [Fre66]. □

In other words, he showed that $\sigma_n(A + B) \geq \sigma_n(B) + \sigma_n(A + B)\sigma_n(A)$, which is stronger than Theorem 3.27 when $\sigma_n(A + B) \geq 1 - \sigma_n(B)$.

The generalization of Mann's Theorem with σ_n seems to still be an open question.

Conjecture 3.29. *Let $A, B \subseteq Q_n$. We have $\sigma_n(A + B) \geq \min(1, \sigma_n(A) + \sigma_n(B))$.*

Although this definition of density seems to be the one with the highest likelihood to be a proper generalization of Mann's Theorem, there is another definition of density, stronger than Definition 3.14, but weaker than Definition 3.25.

In [KK66], Betty Kvarda and R. Killgrove defined the density on *rays*. Let $t = (t_1, \dots, t_n) \in Q_n$ such that $\gcd(t_1, \dots, t_n) = 1$. We define the set $T = \{kt = (kt_1, \dots, kt_n) : k \in \mathbb{N}^*\}$. This set T is called the *ray set* generated by t. We may also define the set $T_j = \{kt = (kt_1, \dots, kt_n) : 1 \leq k \leq j\} \subseteq T$. We are now ready to define the density on rays.

Definition 3.30. Let $A \subseteq Q_n$. We define the *ray density* of A as

$$\sigma_T(A) = \inf_T \inf_{j \in \mathbb{N}^*} \frac{|A \cap T_j|}{|T_j|},$$

where T ranges over all ray set $T \subseteq Q_n$.

This is somewhat *one-dimensional in all directions.* The rays have the nice property that if $a \in A \cap T$ and $b \in B \cap T$, then $a, b, a + b \in (A + B) \cap T$. Using this Kvarda–Killgrove showed the generalization of Mann's Theorem to ray density.

Theorem 3.31. *Let $A, B \subseteq Q_n$. We have $\sigma_T(A + B) \geq \min(1, \sigma_T(A) + \sigma_T(B))$.*

Proof. The proof is available in [KK66]. □

3.4 Conclusion

While Kneser's Theorem and Mann's Theorem describe nicely the growth of sumsets in $\mathbb{N}$, they rely on the linear ordering of the integers, which cannot directly be applied when studying $\mathbb{N}^n$.

Furthermore, the "obvious" ways to generalize the Schnirelmann density fails due to the specific structure on $\mathbb{N}^2$ regardless of the constraints we force onto them. This is particularly evident in Lemma 3.10 and Corollary 3.13. Even the nicer Definition 3.20 (Cheo density) fails to prove the generalization of Mann's Theorem as shown in Example 3.23.

Proved generalizations exist. We have for instance the *fiber density* Proposition 3.16 or the *weighted version* Question 3.17. Both of them are trivially true in the sense that they are direct, once Theorem 3.4 (Mann's Theorem) is established. A more refined generalization is the *ray density*, Theorem 3.31, shown by Kvarda–Killgrove in [KK66]. Despite its ingenuity of looking individually at every ray, it still does not capture the essence of a proper generalization of Mann's Theorem.

The most satisfying generalization of Mann's Theorem corresponds to Conjecture 3.29. Indeed, it nicely extends the Schnirelmann density while still preserving the generalizations of Theorem 3.2 (Schnirelmann's Theorem A) and Theorem 3.3 (Schnirelmann's Theorem B) written in Theorem 3.27 and Theorem 3.26 respectively.

On top of this, no counterexamples to Conjecture 3.29 are known and simulations suggest that no small counterexample exists. It would thus be very reasonable to pursue this path further, trying to prove Conjecture 3.29.

As the last results in this area date back to the 1960s, it is possible that applying modern combinatorial tools may help proving Conjecture 3.29.

Until then, the full generalization remains open...

A Appendix

Lemma A.1. *Let $\theta = 0$ or 1, $0 < \eta \leq 1$, and n a positive integer. Let $A, B \subseteq \mathbb{N} \cap [0, n]$ each containing 0. Then we have that*

$$[\theta, m] \text{ is } (A \vee B)\text{-good with respect to } \eta \text{ for } m = \theta, \theta + 1, \ldots, n, \tag{19}$$

implies

$$[\theta, m] \text{ is } (A + B)\text{-good with respect to } \eta \text{ for } m = \theta, \theta + 1, \ldots, n, \tag{20}$$

Proof. In what follows, when we say something is good, we mean good with respect to η. By way of contradiction, suppose that there exist η and n such that Lemma A.1 is false. We can thus assume that if (19) is true, (20) is false, and that B contains a positive element. We now choose $A, B \subseteq \mathbb{N} \cap [0, n]$ such that (19) is true, (20) is false, and such that $B(0, n)$ is minimal. We will remove certain positive elements from B and placing additional elements in A to arrive at a contradiction. We want to have for the new sets A' and B' :

(i) $[\theta, m]$ is $A' \vee B'$-good for $m = \theta, \theta + 1, \ldots, n$.

(ii) $A' + B' \subseteq A + B$.

(iii) $B'(0, n) < B(0, n)$.

Properties 1 and 2 implies that A' and B' satisfy (19) but not (20) and property 3 contradicts the minimal property of B.

Now, let a^* be the the smallest element $a \in A$ such that $a + B \not\subseteq A$. We are garanteed that such an a^* exists as B contains a positive element and the greatest element of $A + B$ is not in A. Suppose now that if $a^* > 0$ and r is any integer satisfying $0 \leq r < a^*$, we have

$$b + A \cap [0, r] \subseteq A \text{ for every } b \in B, \tag{21}$$

and

$$\text{if } [0, r] \text{ is } A\text{-good, then so is } [b, b + r] \text{ for every } b \in B. \tag{22}$$

We then show that if $a^* > 0$ then for every r satisfying $0 \leq r < a^*$, the interval $[0, r]$ is A-good.

We proceed by way of contradiction. Let $r \in \mathbb{N}$ such that $0 \leq r < a^*$ be the least r for which $[0, r]$ is not A-good. Since $0 \in A$, we see that $r \geq 1$ and that $[1, r]$ is as well not A-good. Thus, $[\theta, r]$ is not A-good. However, we see that $[r, \theta]$ is $A \vee B$-good by (19). This forces B to contain a positive element $b_0 \in [\theta, r]$. As r is minimal, we know $[0, b_0 - 1]$ is A-good. This also shows that the interval $[b_0, r]$ is not A-good. We can now use (21) to show that $[0, r - b_0]$ cannot be A-good, which contradicts the minimality of r.

We now define $B'' = \{b : b \in B, a^* + b \notin A\}$. B'' is non-empty by definition of a^*. We will derive A' and B'. First we define $B' = B \backslash B''$, and then $A' = (A \cup (a^* + B)) \cap [0, n]$. We note that $0 \in B'$ and that $a^* + B' \subseteq A$. We see that B' satisfies (iii). For (ii), we want to show $A' + B' \subseteq A + B$. Clearly $A + B' \subseteq A + B$, so we're left to show $(a^* + B'') + B' \subseteq A + B$. We write one of these elements as

$$(a^* + b'') + b' = (a^* + b') + b'' = a + b'' \in A + B.$$

We finish by showing (i). If $a^* = 0$, we have that $A \vee B = A' \vee B'$, thus the statements (19) and (i) are identical.

We are left with the case $a^* > 0$. Let us consider (i) for a particular m. In this case $A' \vee B'$ and $A \vee B$ differ for the element of B'' by a^* (within $[0, n]$). We need to show that for each m, the interval $[\theta, m]$, remains $A \vee B$-good even if all the positive elements of B in the interval $[m - a^* + 1, m]$ are removed from B to stay in $[0, n]$. Now, let b_1 be the smallest positive $b \in B \cap [m - a^* + 1, m]$ (if the set is empty there is nothing to prove). Take m to be $b_1 + r_1$, with $0 \leq r_1 < a^*$. The interval $[\theta, b_1 - 1]$ is $A \vee B$-good, this reduces the problem to showing that $[b_1, m]$ is A-good. However, by (22), $[0, r]$ is A-good and thus we have $[b_1, b_1 + r] = [b_1, m]$ is A-good which shows (i) and completes the proof of Lemma A.1. □

B Appendix

```
#include <bits/stdc++.h>
using namespace std;
int Rx = 4, Ry = 3;

array<int,2> d(vector<array<int,2>>& s) { //calculates density
    array<int,2> mn = {1,1};
    for (int i=0;i<=Rx;i++) {
        for (int j=0;j<=Ry;j++) {
            if (i == 0 and j == 0) continue;
            int cnt = -1; // do not want to count 0
            for (auto [x,y] : s) if (x <= i and y <= j) cnt++;
            if (cnt*mn[1] <= mn[0]*(i*j + i + j)) // cnt/(i*j+i+j) <= mn[0]/mn[1]
                mn = {cnt,i*j + i + j};
        }
    }
    return mn;
}

vector<array<int,2>> sum(vector<array<int,2>>& A, vector<array<int,2>>& B) {
    set<array<int,2>> CS = {};
    vector<array<int,2>> C = {};
    for (auto [x,y] : A) {
        for (auto [a,b] : B) {
            CS.insert({a+x,b+y});
        }
    }
    for (auto u: CS) C.push_back(u);
    return C;
}

int main() {
    vector<array<int,2>> A = {{0,0},{1,0},{0,1},{3,3}};
    array<int,2> dA = d(A);
    int searchB = 1<<((Rx+1)*(Ry+1));
    for (int i=1; i<searchB;i++) {
        int j = i, index = 1;
        vector<array<int,2>> B = {{0,0}};
        while (j>0) {
            if (j&1) B.push_back({index%(Rx+1),(index/(Rx+1))});
            j /= 2; index++;
        }
        array<int,2> dB = d(B);
        vector<array<int,2>> C = sum(A,B);
        array<int,2> dC = d(C);
        if (dC[0]*dA[1]*dB[1] < (dA[0]*dB[1]+dB[0]*dA[1])*dC[1] and dC[0] < dC[1]) { //checks if dC < dA + dB && dC < 1
            cout << dA[0] << "/" << dA[1] << " + " << dB[0] << "/" << dB[1] << " vs " << dC[0] << "/" << dC[1] << "\n";
            for (auto [x,y] : B) cout << "(" << x << "," << y << ")" << ", ";
            cout << "\n";
            for (auto [x,y] : C) cout << "(" << x << "," << y << ")" << ", ";
            cout << "\n";
            break;
        }
    }
}
```

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