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Understanding the Burgess bound via a trivial delta method

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Contents

1 Introduction to Modular forms and L-function

The goal of this paper is to study and understand [AHLS20] which establishes subconvexity bounds for twisted L-functions. For this, we need to learn about a few concepts. We will start by talking about lattices, then we will define modular forms. We will then continue on L-function of modular forms. Finally, we will look at twists of L-functions and some examples.

1.1 Lattices and Modular Symmetry

We will approach modular forms from a geometric perspective. A lattice Λ in $\mathbb{C}$ is a discrete subgroup generated by two complex numbers ω_1 and ω_2 that are linearly independent over the real numbers. In other words,

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

with $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent in $\mathbb{R}$.

Geometrically, this lattice corresponds to all the points in the complex plane you get by integer linear combinations of the two basis vectors ω_1 and ω_2. The points form a repeating grid, like the corners of parallelograms tiled across the plane.

However, the choice of generators ω_1, ω_2 is not unique. In fact, two pairs (ω_1, ω_2) and (υ_1, υ_2) generate the same lattice if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}),$$

with determinant ± 1, such that

$$\begin{cases} \upsilon_1 = a\omega_1 + b\omega_2, \\ \upsilon_2 = c\omega_1 + d\omega_2. \end{cases}$$

We often want to consider lattices up to scaling, meaning that Λ and $\lambda\Lambda$ for $\lambda \in \mathbb{C}^\star$ are considered equivalent since they represent the same complex torus up to isomorphism.

To understand lattices up to this equivalence, it is common to fix one generator and write $\tau = \omega_1/\omega_2$, which lives in the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

Why the upper half-plane? Because we want the lattice generators to form a positively oriented basis, which translates into the imaginary part of τ being positive.

Now, the group $\mathrm{SL}_2(\mathbb{Z})$, consisting of integer 2×2 matrices with determinant 1,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}$$

acts on $\mathbb{H}$ by

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

for $\gamma \in \mathbb{H}$. This action encodes how changing the lattice basis changes the parameter τ. Two points in $\mathbb{H}$ that lie in the same orbit under this group correspond to lattices that differ by a change of basis in $\mathrm{SL}_2(\mathbb{Z})$.

1.2 Modular Forms: Definitions and Properties

Let $k \in \mathbb{N}$. A modular form of weight k (for the full modular group $\mathrm{SL}_2(\mathbb{Z})$) is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies:

1. Modularity: For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

2. Holomorphicity at the cusp: The function f remains bounded as $\tau \to i\infty$, or equivalently, the function

$$f(\tau) = \sum_{n=0}^{\infty} \lambda_f(n) e^{2\pi i n\tau} = \sum_{n=0}^{\infty} \lambda_f(n) q^n, \quad \text{with } q = e^{2\pi i\tau},$$

has a Fourier expansion with no negative power of q.

The space of such functions is denoted $M_k(\mathrm{SL}_2(\mathbb{Z}))$, or sometimes simply M_k.

A modular form $f \in M_k$ is called a cusp form if it vanishes at the cusp, meaning the constant term in its q-expansion is zero:

$$f(\tau) = \sum_{n=1}^{\infty} \lambda_f(n) q^n.$$

The space of cusp forms is denoted $S_k \subset M_k$. One can see that cusp forms decay rapidly as $\mathrm{Im}(\tau) \to \infty$.

One example of modular form is the Eisenstein Series. For even integers $k \geqslant 4$, the Eisenstein series of weight k is defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n\tau},$$

where B_k is the k-th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of the $(k-1)$-th powers of the divisors of n. Each $E_k(\tau)$ is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. However, it is *not* a cusp form, since it does not vanish at the cusp ∞, in fact, its constant term is 1.

Another such example is the Ramanujan tau function. It is defined as

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \quad q = e^{2\pi i\tau}.$$

This is a cusp form of weight 12.

1.3 Fourier Expansions and Twisted L-functions

Let $f(\tau)$ be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Since f is invariant under $\tau \mapsto \tau + 1$, it admits a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} \lambda_f(n) q^n, \quad q = e^{2\pi i\tau},$$

with $\lambda_f(n) \in \mathbb{C}$ and $\lambda_f(0) = 0$ if f is a cusp form. One associates to f a Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

which converges absolutely for $\mathrm{Re}(s)$ sufficiently large. When f is a normalised Hecke eigenform, the Fourier coefficients $\lambda_f(n)$ satisfy multiplicativity relations $\lambda_f(mn) = \lambda_f(m)\lambda_f(n)$ for $(m, n) = 1$ and $\lambda_f(p^r) = \lambda_f(p)\lambda_f(p^{r-1}) - p^{k-1}\lambda_f(p^{r-2})$ for primes p.

These relations imply that $L(s, f)$ admits an Euler product expansion:

$$L(s, f) = \prod_p \left(1 - \lambda_f(p)p^{-s} + p^{k-1-2s}\right)^{-1},$$

which is analogous to the Euler product for the Riemann zeta function.

The L-function of a cusp form extends to an entire function and satisfies a functional equation relating s to $k-s$. It is natural to then extent this to a twisted version using Dirichlet characters.

Let χ be a Dirichlet character modulo M. The *twisted L-function* is defined by

$$L(s, f \otimes \chi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}.$$

This series converges for $\mathrm{Re}(s)$ large enough and extends to a meromorphic (in fact, entire if f is a cusp form and χ is primitive) function. The twisted L-function has also an Euler product:

$$L(s, f \otimes \chi) = \prod_p \left(1 - \chi(p)\lambda_f(p)p^{-s} + \chi(p)^2 p^{k-1-2s}\right)^{-1},$$

provided f is a Hecke eigenform and χ is primitive.

1.4 Burgess bounds

Understanding the size and distribution of twisted L-values, as well as bounding them in critical regions, is a very interesting topic of study.

A central problem in this direction is to bound $L(s, g \otimes \chi)$ for a fixed Hecke cusp form g and varying Dirichlet character $\chi \pmod{M}$, especially at points on the critical line such as $s = 1/2$. The trivial or convexity bound gives $L(1/2, f \otimes \chi) \ll M^{1/2+\varepsilon}$, but we may want stronger bounds, known as subconvexity bounds.

One of the most influential results in this direction is the *Burgess bound*, originally established in the case of Dirichlet characters. It shows that nontrivial cancellation in character sums leads to an estimate of the form

$$L(1/2, \chi) \ll_\varepsilon M^{1/4-1/16+\varepsilon}.$$

A natural question, is whether one can apply such Burgess bound for L-function of a Hecke cusp form g twisted by a primitive Dirichlet character χ of large conductor M. Indeed, the main theorem we wish to prove in [AHLS20] goes as follow:

Theorem 1.1. *Let g be a fixed Hecke cusp form for $\mathrm{SL}(2,\mathbb{Z})$ and χ be a primitive Dirichlet character modulo a prime M. For any $\varepsilon > 0$,*

$$L\left(\frac{1}{2}, g \otimes \chi\right) \ll_{g,\varepsilon} M^{1/2-1/8+\varepsilon}.$$

We now prove this theorem according to [AHLS20].

2 Some Notations and Lemmas

We start by defining the basics that we will use throughout the proof.

Firstly, for a smooth V with bounded derivatives, or equivalently $V^{(j)} \ll_j 1$, its Fourier transform is defined by

$$\widehat{V}(x) = \int_{\mathbb{R}} V(u)e(-xu)du.$$

Where $e(x) = e^{2\pi i x}$. Repeated integration by parts gives us

$$\widehat{V}(x) \ll_A (1 + |x|)^A, \tag{1}$$

for any $A \geqslant 0$.

Then we have a look at what is called the *Trivial delta method.*

Lemma 2.1. *Whenever one has $q > |n|$ we get*

$$\delta(n=0) = \frac{1}{q}\sum_{c|q}\sum_{\substack{a \bmod c \\ (a,c)=1}} e(\frac{an}{c})$$

Proof. We first notice that every number $n \bmod q$ can be uniquely written as $a \cdot b$ with $a \mid q$ and $b \nmid \frac{a}{q}$. Indeed b is uniquely determined by a and taking $a = (n,q)$ shows existance as $b \mid \frac{(n,q)}{q}$ would imply $ab \neq q$ and uniqueness comes from the fact that if $a < (n,q)$ then we must have $b \mid \frac{(n,q)}{q}$ which is a contradiction. From this, we write

$$\begin{aligned}\delta(n=0) = \delta(n \equiv 0 \pmod q) &= \frac{1}{q}\sum_{a \bmod q} e(\frac{an}{q}) \\ &= \frac{1}{q}\sum_{k|q}\sum_{\substack{a \bmod \frac{q}{k} \\ (a,\frac{q}{k})=1}} e(\frac{akn}{q}) = \frac{1}{q}\sum_{k|q}\sum_{\substack{a \bmod c \\ (a,c)=1}} e(\frac{an}{c}) = \frac{1}{q}\sum_{c|q}\sum_{\substack{a \bmod c \\ (a,c)=1}} e(\frac{an}{c}),\end{aligned}$$

by setting $c = \frac{q}{k}$. □

From this, we observe that $\delta(n \equiv m \bmod q) = \delta(n - m \equiv 0 \bmod q)$ and hence

$$\delta(n \equiv m \bmod q) = \frac{1}{q}\sum_{c|q}\sideset{}{^\star}\sum_{a \bmod c} e(\frac{a(n-m)}{c}).$$

Where $\sum^\star_{a \bmod c}$ is the sum over $(a,c) = 1$.

We continue by defining a very useful formula.

Lemma 2.2. *(Voronoi summation formula, [KMV02, Theorem A.4]) Let g be a holomorphic Hecke cusp form of level 1 and weight k with Fourier coefficients $\lambda_g(n)$. Let $c \in \mathbb{N}$ and $a \in \mathbb{Z}$ be such that $(a,c) = 1$ and let W be a smooth compactly supported function. For $N > 0$,*

$$\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{an}{c}\right)W\left(\frac{n}{N}\right) = \frac{N}{c}\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{-\bar{a}n}{c}\right)\breve{W}_g\left(\frac{nN}{c^2}\right).$$

Here, $\breve{W}_g$ is an integral transform of W given by

$$\breve{W}_g(y) = \int_0^\infty W(x)2\pi i^k J_{k-1}(4\pi\sqrt{yx})dx,$$

where J_k is the Bessel function. We as well have, for any $A \geqslant 0$,

$$\breve{W}_g(x) \ll_A (1+|x|)^{-A}.$$

Another very useful bound to take care of Fourier coefficients is:

Lemma 2.3. *(Ramanujan bound on average [Mol02]) Let W be a smooth function with compact support contained in $R_{\geqslant 0}$, satisfying $W^{(j)} \ll_j 1$. Then*

$$\sum_{n=1}^{\infty}|\lambda_g(n)|W\left(\frac{n}{X}\right) \ll_g X. \tag{2}$$

3 The set up

To start the proof, we begin by defining for $N \geqslant 1$,

$$S(N) = \sum_{n=1}^{\infty}\lambda_g(n)\chi(n)W\left(\frac{n}{N}\right), \tag{3}$$

where W is a smooth bump function supported on $[1,2]$ with bounded derivatives. Using Ramanujan bound on average, one can already deduce $S(N) \ll N^{1+\varepsilon}$.

Remark 3.1. *We note that for what follows, whenever we reference ε, we talk about an arbitrarily small quantity. This ε can change by a constant factor from line to line. As we only have a finite number of operations, ε will still be arbitrarily small.*

This means that if we have for instance $L = M^{1/2}$, then we can write $L^\varepsilon = M^\varepsilon$, as this is just replacing ε by 2ε. This will allow us to best follow [AHLS20].

We now derive the following.

Lemma 3.1. *For any $0 < \delta < 1$, we have*

$$L\left(\frac{1}{2}, g \otimes \chi\right) \ll M^\varepsilon \sup_{N \in (M^{1-\delta}, M^{1+\varepsilon})} \frac{|S(N)|}{\sqrt{N}} + M^{1/2-\delta/2+\varepsilon},$$

where W is a smooth bump function supported on $[1,2]$ with bounded derivatives.

Proof. From the approximate functional equation formula [IK04], we get

$$L\left(\frac{1}{2}, g \otimes \chi\right) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)\chi(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{M}\right) + \varepsilon(g,s) \sum_{n=1}^{\infty} \frac{\overline{\lambda_g(n)\chi(n)}}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{M}\right)$$

We now partition the sum dyadically. Let ψ_N be a smooth function supported on $[N/2, 3N]$ with $\psi_N(n) = 1$ for $n \in [N, 2N]$, and satisfying uniform bounds on its derivatives.

Then,

$$L\left(\tfrac{1}{2}, g \otimes \chi\right) \ll \sum_N \sum_{n=1}^{\infty} \frac{\lambda_g(n)\chi(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{M}\right) \psi_N(n),$$

where the sum over N runs over powers of 2 in the range $M^{1-\delta} < N < M^{1+\varepsilon}$. We can bound the inner sum by

$$\sum_{n=1}^{\infty} \frac{\lambda_g(n)\chi(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{M}\right) \psi_N(n) \ll \frac{1}{\sqrt{N}} |S(N)|.$$

And, as there is at most $O(\log M)$ many dyadic values of N, we can bound the sum by

$$L\left(\frac{1}{2}, g \otimes \chi\right) \ll \sum_{M^{1-\delta} < N < M^{1+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} \ll M^\varepsilon \sup_{M^{1-\delta} < N < M^{1+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} + M^{1/2-\delta/2+\varepsilon}.$$

From one side bounding trivially when $n \leqslant M^{1-\delta}$ and when $n \geqslant M^{1+\varepsilon}$, we use the rapid decay of V to have arbitrarily power saving in M. □

We now define $\mathcal{L}$ to be the set of primes l in the dyadic interval $[L, 2L]$, where $L < M^{1-\varepsilon}$ is a parameter to be determined later. We then denote $L^\star = \sum_{l \in \mathcal{L}} |\lambda_g(l)|^2$. One can see that

$$L^\star = \sum_{l \in \mathcal{L}} |\lambda_g(l)|^2 \asymp \frac{1}{\log L} \sum_{l \in \mathcal{L}} |\lambda_g(l)|^2 \log l = \frac{1}{\log L} \sum_{n=L}^{2L} \Lambda(n) |\lambda_g(n)|^2 - \frac{1}{\log L} \sum_{\substack{n = p^k \in [L, 2L] \\ k \geqslant 2}} |\lambda_g(n)|^2 \log p,$$

and we observe that the error term

$$\frac{1}{\log L} \sum_{\substack{n = p^k \in [L, 2L] \\ k \geqslant 2}} |\lambda_g(n)|^2 \log p \ll \frac{1}{\log L} \sum_{\substack{n = p^k \in [L, 2L] \\ k \geqslant 2}} \log p \ll O(L^{1-\varepsilon}).$$

Using the prime number theorem for automorphic representations [LWY05, Corollary 1.2], one gets

$$L^\star \asymp \frac{1}{\log L} \sum_{n=L}^{2L} \Lambda(n) |\lambda_g(n)|^2 + O(L^{1-\varepsilon}) \asymp \frac{L}{\log L}.$$

We can then proceed similarly for a parameter P and the set $\mathcal{P}$ of its primes p in the dyadic inteval $[P, 2P]$ to get $P^\star = \sum_{p \in \mathcal{P}} 1 \asymp \frac{P}{\log P}$. We choose P and L such that $\mathcal{P} \cap \mathcal{L} = \varnothing$.

Let $p \in \mathcal{P}$, $n \asymp NL$, and $r \asymp N$. For $\varepsilon > 0$ and $PM \gg (NL)^{1+\varepsilon}$, the condition $n = rl$ is equivalent to the congruence $n \equiv rl \mod pM$ as $rl < pM$. Since $N < M^{1+\varepsilon}$, we assume that

$$P \gg L^{1+\varepsilon}. \tag{4}$$

The main sum of interests $S(N)$ can be expressed as

$$S(N) = \sum_{n=1}^{\infty} \chi(n)\lambda_g(n)W\left(\frac{n}{N}\right) = \frac{1}{L^\star}\sum_{l\in\mathcal{L}}\sum_{n=1}^{\infty} \chi(n)\lambda_g(n)W\left(\frac{n}{N}\right)$$

Using the fact that

$$1 = \sum_{r=1}^{\infty} V\left(\frac{r}{N}\right)\delta(n=rl)\left(\frac{n}{rl}\right)^{iv},$$

where V is a smooth function supported on $[1/2, 3]$, we can rewrite

$$\begin{aligned} S(N) &= \frac{1}{L^\star}\sum_{l\in\mathcal{L}}\sum_{n=1}^{\infty}\lambda_g(n)\chi(n)W\left(\frac{n}{N}\right)\sum_{r=1}^{\infty}V\left(\frac{r}{N}\right)\delta(n=rl)\left(\frac{n}{rl}\right)^{iv} \\ &= \frac{1}{L^\star}\sum_{l\in\mathcal{L}}\sum_{r=1}^{\infty}\chi(r)V\left(\frac{r}{N}\right)\sum_{n=1}^{\infty}\lambda_g(n)\delta(n=rl)W\left(\frac{n}{N}\right)\left(\frac{n}{rl}\right)^{iv}. \end{aligned}$$

Using the Hecke relation, we get

$$S(N) = \frac{1}{L^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}\sum_{n=1}^{\infty}\lambda_g(n)W\left(\frac{n}{Nl}\right)\sum_{r=1}^{\infty}\chi(r)V\left(\frac{r}{N}\right)\delta(n=rl)\left(\frac{n}{rl}\right)^{iv} + O(\frac{N^{1+\varepsilon}}{L}).$$

We finish by applying the trivial delta method with $q = pM$ to get

$$\begin{aligned} S(N) = \frac{1}{L^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\sideset{}{^\star}\sum_{\alpha \bmod c}\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{\alpha n}{c}\right)\left(\frac{n}{Nl}\right)^{iv}W\left(\frac{n}{Nl}\right) \\ \times\sum_{r=1}^{\infty}\chi(r)e\left(\frac{-\alpha rl}{c}\right)\left(\frac{r}{N}\right)^{-iv}V\left(\frac{r}{N}\right) + O\left(\frac{N^{1+\varepsilon}}{L}\right). \end{aligned}$$

Here, adding the $V\left(\frac{r}{N}\right)$ allows us to control the bound on N smoothly which will be useful later. Indeed, V here is a smooth function supported on $[1/2, 3]$ constantly 1 on $[1, 2]$ and with bounded derivatives. The $O(N^{1+\varepsilon}L^{-1})$ error term comes from the Hecke relation

$$\lambda_g(rl) = \lambda_g(r)\lambda_g(l) - \delta_{l|r}\lambda_g(1)\lambda_g(r/l).$$

Additionally $(n/rl)^{iv}$ is essentially just multiplying by one and will allow us to control, in the future, the poisson summation formula. Here

$$v := M^\varepsilon. \tag{5}$$

The next step that we would like to take is to apply dual summation formulas to the n and r-sums, followed by applications of Cauchy-Schwarz inequality and Poisson summation to the n-sum.

4 Application of dual summation formulas

Let us come back to our expression of $S(N)$ from the set-up. Namely,

$$\begin{aligned} S(N) = \frac{1}{L^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\sideset{}{^\star}\sum_{\alpha \bmod c}\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{\alpha n}{c}\right)\left(\frac{n}{Nl}\right)^{iv}W\left(\frac{n}{Nl}\right) \\ \times\sum_{r=1}^{\infty}\chi(r)e\left(\frac{-\alpha rl}{c}\right)\left(\frac{r}{N}\right)^{-iv}V\left(\frac{r}{N}\right) + O\left(\frac{N^{1+\varepsilon}}{L}\right). \end{aligned} \tag{6}$$

By applying Voronoi summation formula to the n-sum above and using that g is a cusp form we get

$$S(N) = \frac{1}{L^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\sideset{}{^\star}\sum_{\alpha \bmod c}\frac{Nl}{c}\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{-\overline{\alpha}n}{c}\right)\widetilde{W}_{v,g}\left(\frac{nNl}{c^2}\right)$$
$$\times\sum_{r=1}^{\infty}\chi(r)e\left(\frac{-\alpha rl}{c}\right)\left(\frac{r}{N}\right)^{-iv}V\left(\frac{r}{N}\right)+O\left(\frac{N^{1+\varepsilon}}{L}\right)$$
$$= \frac{N}{L^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}l\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\frac{1}{c}\sum_{n=1}^{\infty}\lambda_g(n)\widetilde{W}_{v,g}\left(\frac{nNl}{c^2}\right)$$
$$\times\sum_{r=1}^{\infty}\chi(r)\sideset{}{^\star}\sum_{\alpha \bmod c}e\left(\frac{-\overline{\alpha}n-\alpha rl}{c}\right)\left(\frac{r}{N}\right)^{-iv}V\left(\frac{r}{N}\right)+O\left(\frac{N^{1+\varepsilon}}{L}\right)$$
$$= \frac{N}{L^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}l\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\frac{1}{c}\sum_{n=1}^{\infty}\lambda_g(n)\widetilde{W}_{v,g}\left(\frac{nNl}{c^2}\right)\sum_{r=1}^{\infty}\chi(r)S(rl,n;c)\left(\frac{r}{N}\right)^{-iv}V\left(\frac{r}{N}\right)$$
$$+O\left(\frac{N^{1+\varepsilon}}{L}\right), \tag{7}$$

where $S(rl,n;c) = \sideset{}{^\star}\sum_{\alpha \bmod c} e\left(\frac{-\overline{\alpha}n-\alpha rl}{c}\right)$ and $\widetilde{W}_{v,g}$ denotes the Hankel transform of the function $W_v(y) := y^{iv}W(y)$, defined in (2.2). Using a stationary phase argument (see A.1 for the details), the function $\widetilde{W}_{v,g}(x)$ is negligibly small, unless $x \asymp v^2 = M^{2\varepsilon}$. Hence the n-variable above is supported on $n \asymp c^2v^2/(Nl)$.

Before applying Poisson summation formula to the r-sums, we have to introduce a few notations. For $a,b \in \mathbb{Z}$, we let (a,b) be the greatest common divisor of a and b, $[a,b]$ their least common multiple, and $a_b = a/(a,b)$. Note that if a is squarefree, then $(b,a_b) = 1$ by prime factorisation.

Currently, the r-sum is expressed as

$$\sum_{r=1}^{\infty}\chi(r)e\left(\frac{-\alpha rl}{c}\right)V_v\left(\frac{r}{N}\right),$$

where $V_v(y) := y^{-iv}V(y)$. We will break this sum modulo $[c,M]$,

$$\sum_{r=1}^{\infty}\chi(r)e\left(\frac{-\alpha rl}{c}\right)V_v\left(\frac{r}{N}\right) = \sum_{\beta \bmod [c,M]}\sum_{r\in\mathbb{Z}}\chi(\beta+r[c,M])e\left(\frac{-\alpha(\beta+r[c,M])l}{c}\right)V_v\left(\frac{\beta+r[c,M]}{N}\right)$$
$$= \sum_{\beta \bmod [c,M]}\sum_{r\in\mathbb{Z}}\chi(\beta)e\left(\frac{-\alpha\beta l}{c}\right)V_v\left(\frac{\beta+r[c,M]}{N}\right)$$
$$= \sum_{\beta \bmod [c,M]}\chi(\beta)e\left(\frac{-\alpha\beta l}{c}\right)\sum_{r\in\mathbb{Z}}V_v\left(\frac{\beta+r[c,M]}{N}\right),$$

and apply Poisson summation to get

$$\frac{N}{[c,M]}\sum_{\beta \bmod [c,M]}\chi(\beta)e\left(\frac{-\alpha\beta l}{c}\right)\sum_{r\in\mathbb{Z}}e\left(\frac{r\beta}{[c,M]}\right)\hat{V}_v\left(\frac{rN}{[c,M]}\right).$$

This can be seen as

$$\hat{V}_v\left(\frac{[c,M]r+\beta}{N}\right) = \int_{\mathbb{R}}V_v\left(\frac{[c,M]s+\beta}{N}\right)e(-sr)ds$$
$$= \frac{N}{[c,M]}\int_{\mathbb{R}}V_v\left(s'\right)e\left(\frac{Ns'-\beta}{[c,M]}r\right)ds'$$
$$= \frac{N}{[c,M]}e\left(\frac{\beta r}{[c,M]}\right)\hat{V}_v\left(\frac{Nr}{[c,M]}\right),$$

by the change of variable $s' = \frac{[c,M]s+\beta}{N}$. We rewrite this as

$$\frac{N}{[c,M]}\sum_{r\in\mathbb{Z}}\left(\sum_{\beta \bmod [c,M]}\chi(\beta)e\left(\frac{-\alpha\beta l}{c}\right)e\left(\frac{r\beta}{[c,M]}\right)\right)\widehat{V}_v\left(\frac{rN}{[c,M]}\right),$$

which using the relation $[c,M] = Mc_M$, makes the β-sum

$$\begin{aligned}&\sum_{\beta \bmod M}\chi(\beta)e\left(\frac{(r-\alpha lM_c)\overline{c_M}\beta}{M}\right)\times\sum_{\beta \bmod c_M}e\left(\frac{(r-\alpha lM_c)\overline{M}\beta}{c_M}\right)\\ &=\overline{\chi}\left((r-\alpha lM_c)c_M\right)g_\chi\times c_M\delta(r-\alpha lM_c\equiv 0 \bmod c_M),\end{aligned}$$

where g_χ is the Gauss sum. By repeated integration by parts (1), we can truncate the r-sum whenever $r \ll [c,M]N^\varepsilon/N$, up to a negligible error. We then have

$$\frac{Ng_\chi}{M}\sum_{\substack{|r|\ll[c,M]N^\varepsilon/N\\ r\equiv\alpha lM_c \bmod c_M}}\overline{\chi}((r-\alpha lM_c)\overline{c_M})\widehat{V}_v\left(\frac{rN}{[c,M]}\right)+O(N^{-2025}).$$

Putting this in (7) by expending $S(rl,n;c)$ yields

$$\begin{aligned}S(N)=\frac{N^2g_\chi}{ML^\star P^\star}&\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}l\sum_{p\in\mathcal{P}}\frac{1}{pM}\sum_{c|pM}\frac{1}{c}\sideset{}{^\star}\sum_{\alpha \bmod c}\sum_{n=1}^{\infty}\lambda_g(n)e\left(\frac{-\overline{\alpha}n}{c}\right)\breve{W}_{v,g}\left(\frac{nNl}{c^2}\right)\\ &\times\sum_{\substack{|r|\ll[c,M]N^\varepsilon/N\\ r\equiv\alpha lM_c \bmod c_M}}\overline{\chi}((r-\alpha lM_c)\overline{c_M})\widehat{V}_v\left(\frac{rN}{[c,M]}\right)+O\left(\frac{N^{1+\varepsilon}}{L}\right).\end{aligned}$$

Let us now try to bound this sum when $c = 1,p,M$. When $c = 1$, $\breve{W}_{v,g}$ gives us arbitrarily small contributions. When $c = p$, $\breve{W}_{v,g}$ gives us again small contributions as we will choose P such that

$$P^2 < M^{1-\delta}L. \tag{8}$$

Finally, when $c = M$, we have

$$\begin{aligned}&\frac{N^2g_\chi}{ML^\star P^\star}\sum_{l\in\mathcal{L}}\overline{\lambda_g(l)}l\sum_{p\in\mathcal{P}}\frac{1}{pM^2}\sum_{n=1}^{\infty}\lambda_g(n)\breve{W}_{v,g}\left(\frac{nNl}{M^2}\right)\\ &\quad\times\sum_{|r|\ll MN^\varepsilon/N}\left(\sideset{}{^\star}\sum_{\alpha \bmod M}\overline{\chi}(r-\alpha l)e\left(\frac{-\overline{\alpha}n}{M}\right)\right)\widehat{V}_v\left(\frac{rN}{M}\right)\\ &\ll M^{1/2}\frac{N^2g_\chi\log L\log P}{M^3LP}\cdot\sum_{l\in\mathcal{L}}l^{1+\varepsilon}\sum_{p\in\mathcal{P}}\frac{1}{p}\sum_{n=1}^{M^2/Nl}n^\varepsilon\times\frac{MN^\varepsilon}{N}\\ &\ll M^{1/2}\frac{N^2g_\chi\log L\log^2 P}{M^3LP}\cdot\sum_{l\in\mathcal{L}}l^{1+\varepsilon}\left(\frac{M^2}{Nl}\right)^{1+\varepsilon}\frac{MN^\varepsilon}{N}\\ &\ll M^{1/2}\frac{g_\chi\log L\log^2 P}{LP}\cdot L\cdot M^\varepsilon\\ &\ll (MLP)^\varepsilon\frac{M}{P}\end{aligned}$$

This is obtained using [AHLS20, Lemma A.1] to get $\sideset{}{^\star}\sum_{\alpha \bmod M}\overline{\chi}(r-\alpha l)e\left(\frac{-\overline{\alpha}n}{M}\right) \ll M^{1/2}$ and using that $|g_\chi| = M^{1/2}$. We now have that

$$S(N) = S^\star(N) + O\left((PML)^\varepsilon\left(\frac{M}{P}+\frac{N}{L}\right)\right), \tag{9}$$

under the condition

$$P^2 < M^{1-\delta}L, \tag{10}$$

where

$$S^\star(N) = \frac{N^2 g_\chi}{M^3 L^\star P^\star} \sum_{l\in\mathcal{L}} \overline{\lambda_g(l)} l \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^2} \sum_{n=1}^{\infty} \lambda_g(n) \breve{W}_{v,g}\left(\frac{nNl}{p^2M^2}\right)$$
$$\times \sum_{|r|\ll pMN^\varepsilon/N} \left(\sideset{}{^\star}\sum_{\alpha \bmod pM} \overline{\chi}(r-\alpha l) e\left(\frac{-\overline{\alpha}n}{pM}\right) \delta(r\equiv \alpha l \bmod p) \right) \widehat{V}_v\left(\frac{rN}{pM}\right) + O(N^{-2025}).$$

Using $(p,m)=1$, the sum $\sideset{}{^\star}\sum_{\alpha \bmod pM} \overline{\chi}(r-\alpha l) e\left(\frac{-\overline{\alpha}n}{pM}\right)\delta(r\equiv \alpha l \bmod p)$ factors as

$$\begin{aligned}
&\sideset{}{^\star}\sum_{\alpha \bmod p} \left(\sideset{}{^\star}\sum_{\alpha' \bmod M} \overline{\chi}(r-(\alpha M+\alpha' p)l) e\left(\frac{-\overline{(\alpha M+\alpha' p)}n}{pM}\right) \delta(r\equiv (\alpha M+\alpha' p) l \bmod p) \right)\\
&= \sideset{}{^\star}\sum_{\alpha \bmod p} \left(\sideset{}{^\star}\sum_{\alpha' \bmod M} \overline{\chi}(r-\alpha' pl) e\left(\frac{-\overline{\alpha M}n}{pM}\right) e\left(\frac{-\overline{\alpha' p}n}{pM}\right) \delta(r\equiv \alpha Ml \bmod p) \right)\\
&= \sideset{}{^\star}\sum_{\alpha \bmod p} e\left(\frac{-\overline{\alpha M}n}{pM}\right) \delta(\overline{r}l\equiv \overline{\alpha M} \bmod p) \times \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r-\alpha l) e\left(\frac{-\overline{\alpha}n}{pM}\right)\\
&= e\left(\frac{-\overline{rM}nl}{p}\right) \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r-\alpha l) e\left(\frac{-\overline{\alpha p}n}{M}\right).
\end{aligned}$$

Since we know $(\alpha, pM)=1$ and that $(l,p)=1$ from $\mathcal{P}\cap\mathcal{L}=\varnothing$ we must have $(r,p)=1$, we can then rewrite $S^\star(N)$ as

$$S^\star(N) = \frac{N^2 g_\chi}{M^3 L^\star P^\star} \sum_{n=1}^{\infty} \lambda_g(n) \sum_{l\in\mathcal{L}} \overline{\lambda_g(l)} l \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^2} \sum_{(r,p)=1} e\left(\frac{-\overline{rM}nl}{p}\right) \tag{11}$$
$$\times \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r+\alpha) e\left(\frac{-\overline{\alpha p}nl}{M}\right) \widehat{V}_v\left(\frac{rN}{pM}\right) \breve{W}_{v,g}\left(\frac{nNl}{p^2M^2}\right).$$

Remark 4.1. *If we now try to estimate the sum $S^\star$ directly we would get*

$$S^\star(N) \ll N^\varepsilon \frac{N^2M^{1/2}}{LPM^3} \frac{P^2M^2}{NL} L^2 \frac{1}{P} \frac{PM}{N} M^{1/2} \ll N^\varepsilon PM,$$

which falls short of $O(PM^\eta)$ from the target bound $O(M^{1-\eta})$. In the next section, we will use Cauchy – Schwarz to smooth the n-variable and apply Poisson summation thereafter to obtain extra saving.

5 Cauchy-Schwarz and Poisson Summation

Recall from the previous section that the n-sum is supported by $\breve{W}_{g,v}$ on $n\sim (pMv)^2/(Nl)$. We set

$$\mathcal{N}_0 = \frac{p^2M^2v^2}{NL}. \tag{12}$$

We can there arrive, up to an arbitrarily error, at

$$S^\star(N) \ll \frac{N^{2+\varepsilon}}{L^\star P^\star M^{5/2}} \sum_{n\geqslant 1} |\lambda_g(n)| U\left(\frac{n}{\mathcal{N}_0}\right) \left| \sum_{l\in\mathcal{L}} \overline{\lambda_g(l)} l \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^2} \sum_{(r,p)=1} e\left(\frac{-\overline{rM}nl}{p}\right)\right.$$
$$\left. \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r+\alpha) e\left(\frac{\overline{\alpha p}nl}{M}\right) \widehat{V_v}\left(\frac{rN}{pM}\right) \breve{W}_v\left(\frac{nNl}{p^2M^2}\right) \right|.$$

Here U is a smooth function with compact support contained in $\mathbb{R}_{>0}$. Applying the Cauchy–Schwarz inequality to the n-sum and then using the Ramanujan bound on average gives

$$S^\star(N) \ll (MNL)^\varepsilon \frac{N^2}{LPM^{5/2}} \left(\sum_{n\geqslant 1} |\lambda_g(n)|^2 U\left(\frac{n}{\mathcal{N}_0}\right)\right)^{1/2} \cdot \left(\sum_{n\geqslant 1} U\left(\frac{n}{\mathcal{N}_0}\right) \left| \sum_{l\in\mathcal{L}} \overline{\lambda_g(l)} l \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^2} \right.\right.$$
$$\left.\left. \times \sum_{(r,p)=1} e\left(\frac{-r\overline{M}nl}{p}\right) \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r+\alpha) e\left(\frac{\overline{\alpha p}nl}{M}\right) \widehat{V_v}\left(\frac{rN}{pM}\right) \widecheck{W}_v\left(\frac{nNl}{p^2M^2}\right)\right|^2\right)^{1/2} + N^{-2025}$$
$$\ll (NML)^\varepsilon \frac{N^2}{LPM^{5/2}} \mathcal{N}_0^{1/2} S^\star(N,\mathcal{N}_0)^{1/2} + N^{-2025}, \tag{13}$$

where

$$S^\star(N,\mathcal{N}_0) = \sum_{n\geqslant 1} U\left(\frac{n}{\mathcal{N}_0}\right) \left| \sum_{l\in\mathcal{L}} \overline{\lambda_g(l)} l \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^2} \sum_{(r,p)=1} e\left(\frac{-r\overline{M}nl}{p}\right)\right.$$
$$\left. \times \sideset{}{^\star}\sum_{\alpha \bmod M} \overline{\chi}(r+\alpha) e\left(\frac{\overline{\alpha p}nl}{M}\right) \widehat{V_v}\left(\frac{rN}{pM}\right) \widecheck{W}_v\left(\frac{nNl}{p^2M^2}\right)\right|^2.$$

Opening the square in $S^\star(N,\mathcal{N}_0)$ and switching the order of summations allows us express

$$S^\star(N,\mathcal{N}_0) = \sum_{l_1,l_2\in\mathcal{L}} \overline{\lambda_g(l_1)} l_1 \lambda_g(l_2) l_2 \sum_{p_1,p_2\in\mathcal{P}} \frac{\chi(p_1)\overline{\chi}(p_2)}{(p_1p_2)^2} \sum_{\substack{(r_1,p_1)=1\\(r_2,p_2)=1}} \widehat{V_v}\left(\frac{r_1N}{p_1M}\right) \overline{\widehat{V_v}\left(\frac{r_2N}{p_2M}\right)}$$
$$\times \sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod M\\ \alpha_2 \bmod M}} \overline{\chi}(r_1+\alpha_1)\chi(r_2+\alpha_2) \times \mathbf{T}, \tag{14}$$

writing

$$\mathbf{T} = \sum_{n=1}^{\infty} e\left(\frac{-r_1\overline{M}nl_1}{p_1}\right) e\left(\frac{r_2\overline{M}nl_2}{p_2}\right) e\left(\frac{\overline{\alpha_1p_1}nl_1 - \overline{\alpha_2p_2}nl_2}{M}\right)$$
$$\times U\left(\frac{n}{\mathcal{N}_0}\right) \widecheck{W}_v\left(\frac{nNl_1}{p_1^2M^2}\right) \overline{\widecheck{W}_v\left(\frac{nNl_2}{p_2^2M^2}\right)}.$$

As in section 4, the rapid decay of $\widecheck{W}_v$ allows us to truncate the r_1, r_2-sums at $r_1, r_2 \ll N^\varepsilon PM/N$, at the cost of a negligible error. For smaller values of r_1 and r_2, we will use the trivial bounds $\widehat{V_v}\left(\frac{r_1N}{p_1M}\right), \widehat{V_v}\left(\frac{r_2N}{p_2M}\right) \ll 1$. We start by breaking $\mathbf{T}$'s n-sum modulo p_1p_2M,

$$\mathbf{T} = \sum_{n\in\mathbb{Z}} \sum_{b \bmod p_1p_2M} e\left(\frac{-r_1\overline{M}bl_1}{p_1}\right) e\left(\frac{r_2\overline{M}bl_2}{p_2}\right) e\left(\frac{\overline{\alpha_1p_1}bl_1 - \overline{\alpha_2p_2}bl_2}{M}\right)$$
$$\times U\left(\frac{b+np_1p_2M}{\mathcal{N}_0}\right) \widecheck{W}_v\left(\frac{(b+np_1p_2M)Nl_1}{p_1^2M^2}\right) \overline{\widecheck{W}_v\left(\frac{(b+np_1p_2M)Nl_2}{p_2^2M^2}\right)}.$$

By doing this, we can expect some cancellation from the $e(\cdot)$ when applying Poisson summation formula. Additionally, Poisson summation formula will permit to truncate the sum even more. Indeed,

$$\begin{aligned}\mathbf{T} =& \sum_{b \bmod p_1p_2M} e\left(\frac{-\overline{r_1M}bl_1}{p_1}\right) e\left(\frac{\overline{r_2M}bl_2}{p_2}\right) e\left(\frac{\overline{\alpha_1p_1}bl_1-\overline{\alpha_2p_2}bl_2}{M}\right)\\ &\times \sum_{n\in\mathbb{Z}} U\left(\frac{b+np_1p_2M}{\mathcal{N}_0}\right)\breve{W}_v\left(\frac{(b+np_1p_2M)Nl_1}{p_1^2M^2}\right)\overline{\breve{W}_v\left(\frac{(b+np_1p_2M)Nl_2}{p_2^2M^2}\right)}\\ =& \sum_{b \bmod p_1p_2M} e\left(\frac{-\overline{r_1M}bl_1}{p_1}\right) e\left(\frac{\overline{r_2M}bl_2}{p_2}\right) e\left(\frac{\overline{\alpha_1p_1}bl_1-\overline{\alpha_2p_2}bl_2}{M}\right)\\ &\times \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} U\left(\frac{b+yp_1p_2M}{\mathcal{N}_0}\right)\breve{W}_v\left(\frac{(b+yp_1p_2M)Nl_1}{p_1^2M^2}\right)\overline{\breve{W}_v\left(\frac{(b+yp_1p_2M)Nl_2}{p_2^2M^2}\right)}e^{2\pi i n y}dy.\end{aligned}$$

Applying the change of variable $x = (b + yp_1p_2M)/\mathcal{N}_0$, yields

$$\begin{aligned}\mathbf{T} =& \sum_{b \bmod p_1p_2M} e\left(\frac{-\overline{r_1M}bl_1}{p_1}\right) e\left(\frac{\overline{r_2M}bl_2}{p_2}\right) e\left(\frac{\overline{\alpha_1p_1}bl_1-\overline{\alpha_2p_2}bl_2}{M}\right)\\ &\times \sum_{n\in\mathbb{Z}}\frac{\mathcal{N}_0}{p_1p_2M}\int_{\mathbb{R}} U(x)\breve{W}_v\left(\frac{x\mathcal{N}_0Nl_1}{p_1^2M^2}\right)\overline{\breve{W}_v\left(\frac{x\mathcal{N}_0Nl_2}{p_2^2M^2}\right)}e\left(\frac{b-x\mathcal{N}_0}{p_1p_2M}n\right)dx\\ =& \frac{\mathcal{N}_0}{p_1p_2M}\sum_{n\in\mathbb{Z}}\sum_{b \bmod p_1p_2M} e\left(\frac{-\overline{r_1M}bl_1}{p_1}\right) e\left(\frac{\overline{r_2M}bl_2}{p_2}\right)\\ &\times e\left(\frac{\overline{\alpha_1p_1}bl_1-\overline{\alpha_2p_2}bl_2}{M}\right) e\left(\frac{bn}{p_1p_2M}\right)\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right),\end{aligned}$$

where

$$\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right) := \int_{\mathbb{R}} U(x)\breve{W}_v\left(\frac{x\mathcal{N}_0Nl_1}{p_1^2M^2}\right)\overline{\breve{W}_v\left(\frac{x\mathcal{N}_0Nl_2}{p_2^2M^2}\right)}e\left(\frac{-nx\mathcal{N}_0}{p_1p_2M}\right)dx.$$

By repeatedly integrating by parts, we can see that the integral $\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right)$ gives arbitrarily power saving in N if $|n| \gg N^{\varepsilon}p_1p_2M/\mathcal{N}_0$. Hence, we can truncate the dual n-sum at $|n| \ll N^{1+\varepsilon}L/(Mv^2)$, at the cost of a negligible error. For smaller values of n, we use the direct bound $\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right) \ll 1$. Using $(p_1p_2, M) = 1$, we get

$$\begin{aligned}\mathbf{T} =& \frac{\mathcal{N}_0}{p_1p_2M}\sum_{n\in\mathbb{Z}}\sum_{b_1 \bmod p_1p_2} e\left(\frac{(-\overline{r_1}l_1p_2+\overline{r_2}+n)\overline{M}b_1M}{p_1p_2}\right)\\ &\times \sum_{b_2 \bmod M} e\left(\frac{(\overline{\alpha_1}l_1p_2-\overline{\alpha_2}l_2p_1+n)\overline{p_1p_2}b_2p_1p_2}{M}\right)\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right)\\ =& \frac{\mathcal{N}_0}{p_1p_2M}\sum_{n\in\mathbb{Z}}\sum_{b \bmod p_1p_2} e\left(\frac{(-\overline{r_1}l_1p_2+\overline{r_2}l_2p_1+n)\overline{M}b}{p_1p_2}\right)\\ &\times \sum_{b \bmod M} e\left(\frac{(\overline{\alpha_1}l_1p_2-\overline{\alpha_2}l_2p_1+n)\overline{p_1p_2}b}{M}\right)\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right)\\ =& \mathcal{N}_0 \sum_{\substack{|n|\ll N^{\varepsilon}\frac{p_1p_2M}{\mathcal{N}_0}\\ -\overline{r_1}l_1p_2+\overline{r_2}l_2p_1+n\equiv 0 \bmod p_1p_2\\ (\overline{\alpha_1}l_1p_2+n,M)=1}} \delta(\alpha_2 \equiv l_2p_1\overline{(\overline{\alpha_1}l_1p_2+n)} \bmod M)\mathcal{J}\left(\frac{n\mathcal{N}_0}{p_1p_2M}\right) + O(N^{-2025}).\end{aligned}$$

It is important to notice that $\overline{r_i}$ is the inverse of $r_i \bmod p_i$ (and not $\bmod p_1p_2$). Setting $R = N^{\varepsilon}PM/N$

and substituting this into 14, gives us

$$S^\star(N,\mathcal{N}_0) \ll N^{-2025} + \mathcal{N}_0 \sum_{l_1,l_2\in\mathcal{L}} \left|\overline{\lambda_g(l_1)}l_1\lambda_g(l_2)l_2\right| \sum_{p_1,p_2\in\mathcal{P}} \frac{1}{(p_1p_2)^2} \sum_{\substack{0\neq|r_1|,|r_2|\ll R\\(r_1,p_1)=(r_2,p_2)=1}} \times \sum_{\substack{|n|\ll N^\varepsilon \frac{p_1p_2M}{\mathcal{N}_0}\\ -\overline{r_1}l_1p_2+\overline{r_2}l_2p_1+n\equiv 0 \bmod p_1p_2}} |\mathfrak{C}|$$

where

$$\mathfrak{C} = \sideset{}{^\star}\sum_{\substack{\alpha \bmod M\\(\overline{\alpha}l_1p_2+n,M)=1}} \overline{\chi}(r_1+\alpha)\chi\left(r_2+l_2p_1(\overline{\overline{\alpha}l_1p_2+n})\right). \tag{15}$$

We now separate the sum in two cases, when $l_1\neq l_2$ and when $l_1=l_2$. In the first case, we apply the Cauchy–Schwarz inequality to the l_i-sums. This allows us to get rid of the Fourier coefficients $\lambda_g(l_i)$ by using the Ramanujan bound on average. Then,

$$S^\star(N,\mathcal{N}_0) \ll S_0^\star(N,\mathcal{N}_0) + S_1^\star(N,\mathcal{N}_0) + N^{-2025}$$

with

$$S_0^\star(N,\mathcal{N}_0) \ll \mathcal{N}_0 \sum_{l\in\mathcal{L}} \left|\overline{\lambda_g(l)}l\right|^2 \sum_{p_1,p_2\in\mathcal{P}} \frac{1}{(p_1p_2)^2} \sum_{\substack{0\neq|r_1|,|r_2|\ll R\\(r_1,p_1)=(r_2,p_2)=1}} \sum_{\substack{|n|\ll N^\varepsilon \frac{p_1p_2M}{\mathcal{N}_0}\\ -\overline{r_1}lp_2+\overline{r_2}lp_1+n\equiv 0 \bmod p_1p_2}} |\mathfrak{C}|$$

and

$$S_1^\star(N,\mathcal{N}_0) \ll \mathcal{N}_0 L^\varepsilon\left(L^2\sum_{l_1\neq l_2\in\mathcal{L}} l_1^2l_2^2\left[\sum_{p_1,p_2\in\mathcal{P}} \frac{1}{(p_1p_2)^2} \sum_{\substack{0\neq|r_1|,|r_2|\ll R\\(r_1,p_1)=(r_2,p_2)=1}} \sum_{\substack{|n|\ll N^\varepsilon \frac{p_1p_2M}{\mathcal{N}_0}\\ -\overline{r_1}l_1p_2+\overline{r_2}l_2p_1+n\equiv 0 \bmod p_1p_2}} |\mathfrak{C}|\right]^2\right)^{\frac{1}{2}}.$$

We will then choose $P < M^{1-\delta-\varepsilon}$, so that $R < M$ and thus $(r_1r_2, M) = 1$. We are then only left to count the number of terms satisfying the congruence conditions and bounding the sums.

We will separate the cases. The first (Δ_1 and Σ_1) when $n \equiv 0 \bmod M$ and the second (Δ_2 and Σ_2) when $n \not\equiv 0 \bmod M$. This way we write

$$S_0^\star(N,\mathcal{N}_0) \ll \mathcal{N}_0(\Delta_1+\Delta_2),$$
$$S_1^\star(N,\mathcal{N}_0) \ll \mathcal{N}_0 L^\varepsilon(\Sigma_1+\Sigma_2)^{1/2}.$$

5.1 When M divides n

We will start by proving the following lemma.

Lemma 5.1. *[AHLS20, Lemma A.2] Suppose that $(r_1r_2, M) = 1$. If $M \mid n$, we have*

$$\mathfrak{C} = \chi(\alpha\overline{\beta})R_M(r_2 - r_1\alpha\overline{\beta}) - \chi(r_2r_1),$$

where $R_M(a) = \sum_{z\in\mathbb{F}_M^\times} e(az/M)$ is the Ramanujan sum. If $M \nmid n$ and at least one of $r_1 - \overline{n}\beta$ and $r_2 + \overline{n}\alpha$ is nonzero in $\mathbb{F}_M$, then

$$\mathfrak{C} \ll M^{1/2}.$$

Finally, if $n \neq 0$ and $r_1 - \overline{n}\beta = r_2 + \overline{n}\alpha = 0$ in $\mathbb{F}_M$, then

$$\mathfrak{C} = \begin{cases} -\chi(nr_2\overline{\beta}) & \text{if } \chi \text{ is not a quadratic character,}\\ \chi(\overline{n}r_2\beta)(M-1) & \text{if } \chi \text{ is a quadratic character.}\end{cases}$$

Using this lemma and applying parameters $(\alpha, \beta) = (l_2p_1, l_1p_2)$ gives us

$$\mathfrak{C} = \chi(l_2p_1\overline{l_1p_2}^1)R_M(r_2 - r_1l_2p_1\overline{l_1p_2})\chi(r_2\overline{r_1}) = \begin{cases} O(M), & \text{if } r_2l_1p_2 \equiv r_1l_2p_1 \bmod M \\ O(1), & \text{otherwise.} \end{cases}$$

According to $r_2l_1p_2 \equiv r_1l_2p_1 \bmod M$ or not, we write

$$\Delta_1 = \Delta_{10} + \Delta 11 \quad \text{and} \quad \Sigma_1 = \Sigma_{10} + \Sigma_{11},$$

where

$$\Delta_{10} := \sum_{l\in\mathcal{L}} \left|\overline{\lambda_g(l)l}\right|^2 \sum_{p_1,p_2\in\mathcal{P}} \frac{1}{(p_1p_2)^2} \sum_{\substack{0\neq|r_1|,|r_2|\ll R \\ (r_1,p_1)=(r_2,p_2)=1 \\ r_2p_2\equiv_M r_1p_1}} \sum_{\substack{|n|\ll N^\varepsilon \frac{p_1p_2M}{\mathcal{N}_0} \\ -\overline{r_1}lp_2+\overline{r_2}lp_1+n\equiv 0 \bmod p_1p_2 \\ n\equiv 0 \bmod M}} M,$$

and

$$\Sigma_{10} := L^2 \sum_{l_1\neq l_2\in\mathcal{L}} l_1^2l_2^2 \left[\sum_{p_1,p_2\in\mathcal{P}} \frac{1}{(p_1p_2)^2} \sum_{\substack{0\neq|r_1|,|r_2|\ll R \\ (r_1,p_1)=(r_2,p_2)=1 \\ r_2l_1p_2\equiv_M r_1l_2p_1}} \sum_{\substack{|n|\ll N^\varepsilon \frac{p_1p_2M}{\mathcal{N}_0} \\ -\overline{r_1}l_1p_2+\overline{r_2}l_2p_1+n\equiv 0 \bmod p_1p_2 \\ n\equiv 0 \bmod M}} M \right]^2.$$

Δ_{11} and Σ_{11} are the same sums with the congruence condition $r_2l_1p_2 \neq r_1l_2p_1 \bmod M$ and summing 1s and not Ms. Opening the square, we write Σ_{10} and Σ_{11} as a sum over $l_i, n, n', r_i, r_i', p_i, p_i'$ for $i = 1, 2$. Then, under the assumption

$$P^2L < N^{1-\epsilon}, \tag{16}$$

we show

$$\Sigma_{10} \ll (PML)^\varepsilon \frac{L^6M^4}{P^4N^2} \quad \text{and} \quad \Delta_{10} \ll (PML)^\varepsilon \frac{L^3M^2}{P^2N}. \tag{17}$$

Proof. We recall $|r_2| < R = N^\varepsilon PM/N$, along with $P^2L < N^{1-\varepsilon}$. Thus, $r_2l_1p_2 \equiv r_1l_2p_1 \bmod M$ implies that $r_2l_1p_2 = r_1l_2p_1$. As this is an equality, fixing five variables fixes all six. Additionally, fixing l_1, p_2, r_2 fixes l_2, r_1, p_1 in Δ_{10} up to factors of $\log M$. Indeed, as $l_1 = l_2$, then p_1 must be a prime divisor of r_2, so at most $O(\log M)$ possibilities. When $l_1 \neq l_2$, then the equality $r_2'l_1p_2' = r_1'l_2p_1'$ implies that $l_2|r_2'$, so up to $O(\log M)$ we saved a factor in L. Then, we can proceed as in Δ_{10}.

Additionally, we can combine $-\overline{r_1}l_1p_2 + \overline{r_2}l_2p_1 + n \equiv 0 \bmod p_1p_2$ and $n \equiv 0 \bmod M$ to write

$$-\overline{r_1}l_1p_2 + \overline{r_2}l_2p_1 + n \equiv 0 \bmod p_1p_2M.$$

As $n \ll N^{1+\varepsilon}L/M$ is smaller than p_1p_2M, we know that for fixed l_i, p_i, r_i the n-sum has at most one element, likewise for n'. Thus, we indeed have

$$\Sigma_{10} \ll (PML)^\varepsilon L^2 \cdot L^6 \cdot \frac{RM}{P^3L} \cdot \frac{RM}{P^3L} \ll (PML)^\varepsilon \frac{L^6M^4}{P^4N^2},$$
$$\Delta_{10} \ll (PML)^\varepsilon L^4 \frac{RM}{P^3L} \ll (PML)^\varepsilon \frac{L^3M^2}{P^2N}.$$

□

Additionally, we show

$$\Sigma_{11} \ll (PML)^\varepsilon \frac{L^8M^4}{P^4N^4}\left(1 + \frac{P^2}{\mathcal{N}_0}\right)^2 \quad \text{and} \quad \Delta_{11} \ll (PML)^\varepsilon \frac{L^3M^2}{P^2N^2}\left(1 + \frac{P^2}{\mathcal{N}_0}\right).$$

Proof. We proceed like above. Firstly when $l_1 \neq l_2$. If $p_1 \neq p_2$, then $(n, p_1p_2) = 1$ and using $-\overline{r_1}l_1p_2 + \overline{r_2}l_2p_1 + n \equiv 0 \bmod p_1p_2$ we deduce $r_1 \equiv \overline{n}l_1p_2 \bmod p_1$ and $r_2 = -\overline{n}l_2p_1$. Using $R \gg P$ in the congruences, we save a factor of $O(P)$ in each r_i and r_i' sum. The congruence $n \equiv 0 \bmod M$ saves a factor of $O(M)$ and likewise for the n'-sum. If $p_1 = p_2$, then the congruence conditions forces $p|n$.

[1] In [AHLS20], there is a small typo inverting l_1p_2 with l_2p_1.

We must have $|n| \ll N^{1+\varepsilon}L/M$, but combining $M|n$, $p|n$, and that $N^{1+\varepsilon}L/M < pM$ gives us $n = 0$. We are left with $r_1 l_2 \equiv r_2 l_1 \bmod p_1$, thus fixing r_1, l_2, l_1 saves another factor of $O(P)$ in the r_2 sum. We proceed in the same way for the n', r_2' sums. The case $l_1 = l_2$ follows the same argument. Thus, we have

$$\Sigma_{11} \ll (PML)^{\varepsilon} \left(L^8 \left[\frac{1}{P^2} \frac{R^2}{P^2} \left(1 + \frac{P^2}{\mathcal{N}_0}\right)\right]^2 + L^8 \left[\frac{1}{P^3}\frac{R^2}{P}\right]^2 \right) \ll (PML)^{\varepsilon} \frac{L^8 M^4}{N^4 P^4} \left(1 + \frac{P^2}{\mathcal{N}_0}\right)^2,$$

$$\Delta_{11} \ll (PML)^{\varepsilon} L^3 \left[\frac{1}{P^2}\frac{R^2}{P^2}\left(1 + \frac{P^2}{\mathcal{N}_0}\right)\right] + L^3 \left[\frac{1}{P^2}\frac{R^2}{P^2}\left(1 + \frac{P}{\mathcal{N}_0}\right)\right] \ll (PML)^{\varepsilon} \frac{L^3 M^2}{N^2 P^2}\left(1 + \frac{P^2}{\mathcal{N}_0}\right).$$

□

5.2 When M does not divide n

This time Lemma 5.1 gives us

$$\mathfrak{C} = \begin{cases} O(M), & \text{if } r_1 - \overline{n} l_1 p_2 \equiv r_2 + \overline{n} l_2 p_1 \equiv 0 \bmod M \\ O(M^{1/2}), & \text{otherwise.} \end{cases}$$

Depending if $r_1 - \overline{n} l_1 p_2 \equiv r_2 + \overline{n} l_2 p_1 \equiv 0 \bmod M$ or not, we write

$$\Delta_2 = \Delta_{20} + \Delta_{21} \quad \text{and} \quad \Sigma_2 = \Sigma_{20} + \Sigma_{21},$$

where

$$\Delta_{20} := \sum_{l \in \mathcal{L}} \left|\overline{\lambda_g(l)} l\right|^2 \sum_{p_1, p_2 \in \mathcal{P}} \frac{1}{(p_1 p_2)^2} \sum_{\substack{0 \neq |r_1|, |r_2| \ll R \\ (r_1, p_1) = (r_2, p_2) = 1}} \sum_{\substack{|n| \ll N^{\varepsilon} \frac{p_1 p_2 M}{\mathcal{N}_0} \\ -\overline{r_1} l p_2 + \overline{r_2} l p_1 + n \equiv 0 \bmod p_1 p_2 \\ n \not\equiv 0 \bmod M \\ r_1 - \overline{n} l p_2 \equiv r_2 + \overline{n} l p_1 \equiv 0 \bmod M}} M,$$

and

$$\Sigma_{20} := L^2 \sum_{l_1 \neq l_2 \in \mathcal{L}} l_1^2 l_2^2 \left[\sum_{p_1, p_2 \in \mathcal{P}} \frac{1}{(p_1 p_2)^2} \sum_{\substack{0 \neq |r_1|, |r_2| \ll R \\ (r_1, p_1) = (r_2, p_2) = 1}} \sum_{\substack{|n| \ll N^{\varepsilon} \frac{p_1 p_2 M}{\mathcal{N}_0} \\ -\overline{r_1} l_1 p_2 + \overline{r_2} l_2 p_1 + n \equiv 0 \bmod p_1 p_2 \\ n \not\equiv 0 \bmod M \\ r_1 - \overline{n} l_1 p_2 \equiv r_2 + \overline{n} l_2 p_1 \equiv 0 \bmod M}} M \right]^2.$$

Δ_{21} and Σ_{21} are the other sums summing $M^{1/2}$s and not Ms. Opening the square, we write Σ_{20} and Σ_{21} as a sum over $l_i, n, n', r_i, r_i', p_i, p_i'$ for $i = 1, 2$. Then, we show

$$\Sigma_{20} \ll (PML)^{\varepsilon} \frac{L^6 M^4}{P^4 N^2} \quad \text{and} \quad \Delta_{20} \ll (PML)^{\varepsilon} \frac{L^3 M^2}{P^2 N}.$$

Proof. We start by observing that $|nR| \ll N^{\varepsilon} PL < PM^{1-\varepsilon}$. Thus, when we merge the congruence conditions on r_1 and r_2 to get

$$n r_1 \equiv l_1 p_2 \bmod M \quad \text{and} \quad n r_2 \equiv -l_2 p_1 \bmod M,$$

we actually have an equality. From this, we deduce $l_1 p_2 / r_1 = n = -l_2 p_1 / r_2$, so we have $l_1 p_2 r_2 = -l_2 p_1 r_1$. Therefore, fixing $l_1 p_2 r_2$ fixes the three others up to $O(\log M)$ like in (17). The same applies for r_1' and r_2' so that $l_1 p_2' r_2' = -l_2 p_1' r_1'$. If $l_1 \neq l_2$, then l_2 must divide r_2' which saves a factor of $O(L)$ in the r_2'-sum. Furthermore, fixing l_1, p_2', r_2' fixes p_1' and r_1' up to $O(\log M)$. The case $l_1 = l_2$ makes fixing p_1 and r_1 fix p_2 and r_2 up to $O(\log M)$, and $n r_1 = l p_2$ fixes n. Therefore, we have

$$\Sigma_{20} \ll (PML)^{\varepsilon} L^2 \cdot L^6 \cdot \frac{RM}{P^3 L} \cdot \frac{RM}{P^3 L} \ll (PML)^{\varepsilon} \frac{L^6 M^4}{P^4 N^2},$$

$$\Delta_{20} \ll (PML)^{\varepsilon} L^4 \frac{RM}{P^3 L} \ll (PML)^{\varepsilon} \frac{L^3 M^2}{P^2 N}.$$

□

Additionally, we show that

$$\Sigma_{21} \ll (PML)^{\varepsilon} \frac{L^8 M^4}{P^4 N^4} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right)^2 \quad \text{and} \quad \Delta_{21} \ll (PML)^{\varepsilon} \frac{L^3 M^{5/2}}{P^2 N^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right).$$

Proof. Firstly, if $p - 1 \neq p_2$, the congruence $-\overline{r_1} l_1 p_2 + \overline{r_2} l_2 p_1 + n \equiv 0 \bmod p_1 p_2$ implies $(n, p_1 p_2) = 1$. As in the proof above, we can save a factor of $O(P)$. When $p_1 = p_2$, the congruence condition forces $p || n$. Using $r_2 \equiv \overline{(\overline{r_1} l_1 - n/p)} l_2 \bmod p$, we see that we can save a factor of $O(P)$ in each of the r_2,r_2', and n'-sums. We repeat the same argument for the $l_1 = l_2$ case. Thus, this gives us

$$\begin{aligned}
\Sigma_{21} &\ll (PML)^{\varepsilon} \left(L^8 \left[\frac{1}{P^2} \frac{R^2}{P^2} \left(1 + \frac{P^2}{\mathcal{N}_0}\right) M^{1/2} \right]^2 + L^8 \left[\frac{1}{P^3} \frac{R^2}{P} \left(1 + \frac{PM}{\mathcal{N}_0}\right) M^{1/2} \right]^2 \right) \\
&\ll (PML)^{\varepsilon} \frac{L^8 M^5}{N^4 P^4} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right)^2, \\
\Delta_{21} &\ll (PML)^{\varepsilon} \left(L^3 \left[\frac{1}{P^2} \frac{R^2}{P^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right) M^{1/2} \right] + L^3 \left[\frac{1}{P^2} \frac{R^2}{P^2} \left(1 + \frac{PM}{\mathcal{N}_0}\right) M^{1/2} \right] \right) \\
&\ll (PML)^{\varepsilon} \frac{L^3 M^{5/2}}{N^2 P^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right).
\end{aligned}$$

□

6 Conclusion

The bounds for $\Delta_1, \Delta_2, \Sigma_1, \Sigma_2$ imply

$$\begin{aligned}
S^{\star}(N, \mathcal{N}_0) &\ll \mathcal{N}_0 \left(\Delta_{10} + \Delta_{11} + \Delta_{20} + \Delta_{21} + L^{\varepsilon} (\Sigma_{10} + \Sigma_{11} + \Sigma_{20} + \Sigma_{21})^{1/2} \right) \\
&\ll \mathcal{N}_0 (PML)^{\varepsilon} \left(\frac{L^3 M^2}{P^2 N} + \frac{L^3 M^2}{P^2 N^2} \left(1 + \frac{P^2}{\mathcal{N}_0}\right) + \frac{L^3 M^2}{P^2 N} + \frac{L^3 M^{5/2}}{P^2 N^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right) \right. \\
&\qquad \left. + \left[\frac{L^6 M^4}{P^4 N^2} + \frac{L^8 M^4}{P^4 N^4} \left(1 + \frac{P^2}{\mathcal{N}_0}\right)^2 + \frac{L^6 M^4}{P^4 N^2} + \frac{L^8 M^5}{P^4 N^4} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right)^2 \right]^{1/2} \right) \\
&\ll \mathcal{N}_0 (PML)^{\varepsilon} \left[\frac{L^3 M^2}{P^2 N} + \frac{L^4 M^{5/2}}{P^2 N^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right) \right],
\end{aligned}$$

Inserting this into 13 and using 12 gives us

$$\begin{aligned}
S^{\star}(N) &\ll (NML)^{\varepsilon} \frac{N^2}{LPM^{5/2}} \mathcal{N}_0^{1/2} \cdot \mathcal{N}_0^{1/2} (PML)^{\varepsilon} \left[\frac{L^3 M^2}{P^2 N} + \frac{L^4 M^{5/2}}{P^2 N^2} \left(1 + \frac{P^2 M}{\mathcal{N}_0}\right) \right]^{1/2} \\
&\ll (PML)^{\varepsilon} \frac{PNv^2}{L^2 M^{1/2}} \left[\frac{L^{3/2} M}{PN^{1/2}} + \frac{L^2 M^{5/4}}{PN} + \frac{M^{3/4} L^{5/2}}{PN^{1/2} v} \right] \\
&\ll (PML)^{\varepsilon} \left[\frac{N^{1/2} M^{1/2} v^2}{L^{1/2}} + M^{3/4} v^2 + N^{1/2} L^{1/2} M^{1/4} v \right]
\end{aligned}$$

Recalling that $v = M^{\varepsilon}$ from (5), and using the above into (9) gives us

$$\frac{S(N)}{N^{1/2}} \ll (PML)^{\varepsilon} \left[\frac{N^{1/2} L}{PM^{1/2}} + \frac{M}{PN^{1/2}} + \frac{M^{1/2}}{L^{1/2}} + \frac{M^{3/4}}{N^{1/2}} + L^{1/2} M^{1/4} \right].$$

We now make the choice of $P = M^{1/4+\varepsilon}$ and $L = P^{1-\varepsilon}$. This ensures that the conditions (4), (8) and (16) are satisfied and make the sum

$$\frac{S(N)}{N^{1/2}} \ll M^{\varepsilon} \left(\frac{M^{3/4}}{N^{1/2}} + M^{3/8} \right).$$

Using 3.1, we can take $\delta = 1/4$ which means that $N > M^{3/4}$, and $N \ll M^{3/4}$. We thus obtain

$$L\left(\frac{1}{2}, g \otimes \chi\right) \ll M^{3/8+\varepsilon},$$

as desired, which proves Theorem 1.1.

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A Appendix

Lemma A.1. *(Stationary phase argument [LMS22, Lemma 3.3]) Let $W(y)$ be a smooth, compactly supported function on $\mathbb{R}_{\geqslant 0}$, and remind ourselves that*

$$W_v(y) := y^{iv} W(y),$$

and

$$\widetilde{W}_g(y) = \int_0^\infty W(x) 2\pi i^k J_{k-1}(4\pi\sqrt{yx})dx,$$

where J_g is a Bessel-type kernel arising in the Hankel transform associated to a generic representation of $\mathrm{GL}_3(\mathbb{R})$*. Then for any $A > 0$, we have*

$$\widetilde{W}_{v,g}(x) \ll_A (x+v)^{-A},$$

unless $x \asymp v^2$.

Proof. We want to show that the Hankel transform of $W_{v,g}(y) := y^{iv}W(y)$ is negligibly small unless $x \asymp v^2$. The Hankel transform is given by

$$\widetilde{W}_{v,g}(x) = 2\pi \int_0^\infty y J_{iv}(4\pi\sqrt{xy}) W_{v,g}(y) dy = 2\pi \int_0^\infty y^{1+iv} W(y) J_{iv}(4\pi\sqrt{xy}) dy,$$

where $W(y)$ is a smooth function compactly supported on $y \asymp 1$. For large v, we use the asymptotic expansion of the Bessel function for large order (see [AS64, 9.2.1 p.364]):

$$J_{iv}(z) \approx \frac{1}{\sqrt{2\pi z}} \left(e^{i(z - \frac{\pi}{2}iv - \frac{\pi}{4})} + e^{-i(z - \frac{\pi}{2}iv - \frac{\pi}{4})}\right).$$

Let $z = 4\pi\sqrt{xy}$. The transform becomes a sum of two integrals. The dominant contribution comes from the term whose phase has a stationary point. Let's consider the two phase functions:

$$\varphi_\pm(y) = \pm 4\pi\sqrt{xy} + v \ln y.$$

We look for a stationary point by setting the derivative of the phase with respect to y to zero:

$$\varphi_\pm'(y) = \pm\frac{2\pi\sqrt{x}}{\sqrt{y}} + \frac{v}{y}.$$

Setting $\varphi_+'(y) = 0$ gives no solution for real and positive v, x, y. Setting $\varphi_-'(y) = 0$ at a point y_c gives:

$$-\frac{2\pi\sqrt{x}}{\sqrt{y_c}} + \frac{v}{y_c} = 0 \implies v = 2\pi\sqrt{xy_c}.$$

Since $W(y)$ is supported on $y \asymp 1$, we must have $y_c \asymp 1$ for the integral to be non-negligible. This implies:

$$v \asymp 2\pi\sqrt{x},$$

which simplifies to

$$x \asymp \frac{v^2}{4\pi^2} \quad \text{or simply} \quad x \asymp v^2.$$

If this condition is not met, the phase function has no stationary point in the support of $W(y)$. By the principle of stationary phase, the integral is then rapidly oscillating and thus negligibly small. Therefore, $\breve{W}_{v,g}(x)$ is non-negligible only when $x \asymp v^2$.

□

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